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Statistical-Thermodynamic Approach to a Chaotic Dynamical System: Exactly Solvable Examples

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The static and dynamic properties of a chaotic attractor of a two-dimensional map are studied, which belongs to a particular class of piecewise continuous invertible maps. Coverings of a natural size to cover the attractor are introduced, so that the microscopic information of the attractor is written on each box composing the cover. The statistical thermodynamics of the scaling indices and the size indices of the boxes is formulated. Analytic forms of the free energy functions of the scaling indices and the size indices of the boxes are obtained for examples of a hyperbolic and a nonhyperbolic chaotic attractor. The statistical thermodynamics of local Lyapunov exponents is also studied and a relation between the thermodynamics of scaling indices and of local Lyapunov exponents is invetigated. For the nonhyperbolic example, the free energy and entropy functions of local Lyapunov exponents are obtained in analytic forms. These results display the existence of phase transitions. A phase transition is seen in the thermodynamics of scaling indices also.

KEY WORDS: Chaos; natural measure; scaling index; symbol sequence; one-dimensional lattice system; thermodynamic approach; generalized dimension; local Lyapunov exponent; generalized entropy; nonhyperbolic attractor; phase transition; scaling law.

1. INTRODUCTION

In order to characterize and understand chaos,[1] order parameters in chaos have been sought. Recently, three new quantities (the scaling indices of an invariant measure,[2-5] the time damping rates of a probability measure in trajectory space,[6,7] and local Lyapunov exponents of nearby orbits[8]) were found to characterize chaos, and methods similar to the

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thermodynamic formalisms in equilibrium statistical mechanics have been developed for the calculation of these quantities.[9–12] By using thermodynamic approaches to the new quantities, many studies have been done on strange attractors,[13] such as the attractor of a chaotic dynamical system.[14–29] Such work has given important material for deciding whether or not the new quantities are capable of being order parameters. To be a good order parameter for chaos, a quantity must be capable of describing and characterizing qualitative changes of the attractor at the bifurcation point of a band merging, a crisis, or a saddle-node bifurcation of intermittent chaos. It has been found that the local Lyapunov exponents characterize these bifurcations as the occurrence of nonanalyticity in its free energy function.[25–28] But the correspondence between the bifurcations and the occurrence of nonanalyticity is not complete.[18,28,30] Nonanalytic behaviors have also been found in generalized dimensions[12,27,31] and in generalized entropies[19] on a nonhyperbolic chaotic attractor.[33]

On the other hand, relations between thermodynamic variables associated with the new quatities have also been studied.[12–16,32] For a hyperbolic chaotic attractor of a two-dimensional (2D) map, it is known that the following relations hold[14,32]:

$$\Phi(\tau_1(q), \tau_2(q)) = 0 \tag{1.1}$$

between the free energy function of local Lyapunov exponents $\boldsymbol{\Lambda} \equiv (\Lambda_1, \Lambda_2)$ [defined as in (4.8)] and the generalized partial dimensions $D_j(q) \equiv \tau_j(q)/(q-1)$ $(j = 1, 2)$,[34] and[12,15]

$$\Psi(q) = \Phi(z_1 = q-1, z_2 = 0) \tag{1.2}$$

between the free energy function of $\boldsymbol{\Lambda}$ and the generalized entropies $K(q) \equiv \Psi(q)/(q-1)$.[7] For a 2D map with constant Jacobian, it turns out from (1.1) that the nonanalyticity of the free energy function of $\boldsymbol{\Lambda}$ leads to nonanalyticity in generalized dimensions if (1.1) holds.[27] However, the above relations without any restriction on q may be incorrect for a nonhyperbolic chaotic attractor. It is of interest to see how the nonanalyticity of the free energy function of $\boldsymbol{\Lambda}$ correlates with the validity of (1.1) and (1.2). We do not have a certain rule to determine each direction of the generalized partial dimensions for a nonhyperbolic chaotic attractor, while for a hyperbolic chaotic attractor the generalized partial dimensions are measured at every point on the attractor in the directions tangent to the unstable and stable manifolds, respectively.[13] When the directions of the partial dimensions do not exactly coincide with those of the local Lyapunov exponents, we do not know whether (1.1) and (1.2) hold or not.

Most of the many works using the thermodynamic approach to the new quantities have been done on chaotic attractors in one-dimensional (1D) systems[9,16–23] or numerically on strange attractors in higher-dimensional systems.[24–28] In a 1D dynamical system, a chaotic attractor often has a trivial structure. The 1D system is less useful for understanding fractal structures of a chaotic attractor, but may be useful for understanding singular behaviors of a local Lyapunov exponent on a chaotic attractor.[14,20,26] After all, the 1D systems cannot say anything about the relation (1.1). With the numerical work it is difficult to obtain accurate results and this work is insufficient to provide conclusions, especially about the validity of the relations. We have only a few examples of a 2D chaotic attractor which can be exactly analyzed in terms of the thermodynamic variables.[12,16,29] Therefore, it is desirable to have further examples of such attractors and to examine the relations (1.1) and (1.2) on these attractors.

The thermodynamics of scaling indices gives macroscopic information about the singularity of a natural measure on a chaotic attractor. In the formulation of the thermodynamics a set of small boxes is used to cover the attractor.[3,5] Not much attention has been given to methods of making the covering; in some cases a partition of phase space into a uniform grid is used,[7,15] in other cases a dynamical partition of equal mass,[35] and so on. Singular structures of a strange attractor occur not only in the probability measure, but also in the geometry of the attractor.[1,13] In order to describe the geometrical singularity, Kohmoto[11] introduced the size index of a box and developed the thermodynamics of the scaling index and the size index of a box. In this work, I consider a chaotic dynamical system generated by a 2D map which is invertible and piecewise continuous. In the 2D system, I will extend his formulation to partial variables and expand their statistical thermodynamics concretely. In special examples, including a nonhyperbolic chaotic attractor, I will analyze the statistical thermodynamics of local Lyapunov exponents and examine (1.1) and (1.2).

This paper consists of six sections and two appendices. In Section 2, families of two-dimensional maps are introduced. I give a lemma for a natural measure on a chaotic attractor of the map. Local Lyapunov exponents are given and discussed in relation to an unstable manifold of the chaotic attractor. In Section 3, the statistical thermodynamics of scaling indices of the natural measure is constructed. A sequence of covers of the chaotic attractor is introduced and each element of the cover, i.e., a box, is represented by a symbol sequence. Not only scaling indices (α_x, α_y), but also size indices (β_x, β_y) of a box are defined on the space of a symbol sequence. The free energy and entropy functions of α_ν and β_ν, $\nu = x$ and y, are given. We get a relation between the free energy functions of α_ν and of α_ν and β_ν. The statistical thermodynamics of a time damping rate γ is also given and a relation between the spectra of α and γ is studied. Sections 4 and 5 give detailed calculations of examples, Section 4 for hyperbolic chaotic attractors and Section 5 for the nonhyperbolic case. The thermodynamics of Λ as well as of α_v and β_v is exactly analyzed. In particular, for the nonhyperbolic attractor I give rigorous results for the free energy function $\Phi(z_1, z_2)$ and entropy function $s_A(A_1, A_2)$ of Λ. I also obtain a rigorous result for the free energy function of α_v and β_v. Then, I find phase transitions in the thermodynamics both of Λ and of α_y and β_y. The phase diagram of Λ is described on the $z_1 z_2$ plane. The relations (1.1) and (1.2) are examined and found to break down at phase boundaries, while they hold in some region on the $z_1 z_2$ plane. Zeros of the partition functions are calculated and discussed in relation to the phase transitions. In Section 6, I reexamine (1.1) and discuss why (1.1) holds in some regions and not in others. Finally, I give a summary. In order to obtain $s_A(A_1, A_2)$ in Section 5, Appendix A is useful. Appendix B discusses a scaling form of Λ near the phase transition points.

2. TWO-DIMENSIONAL MAPS

Let us consider the dynamical system generated by the invertible map of the unit square $S = [1, 0] \otimes [0, 1]$ to itself:

$$X_{n+1} = X(x_n, y_n), \qquad y_{n+1} = Y(y_n) \tag{2.1}$$

X is a piecewise continuous function given by

$$X(x, y) = \kappa_\sigma x + t_\sigma(y) \qquad \text{for} \quad (x, y) \in S_\sigma \quad (\sigma = 0, 1) \tag{2.2}$$

where $S_0 = [0, 1] \otimes [0, c)$ and $S_1 = [0, 1] \otimes [c, 1]$, the constants κ_0 and κ_1 are positive, and $t_0(y)$ and $t_1(y)$ are single-valued continuous functions on S_0 and S_1, respectively. Y is given as follows (see Fig. 1).

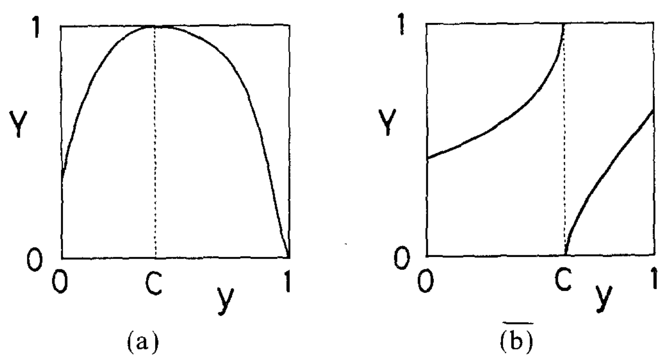


Fig. 1. Examples of the one-dimensional maps (a) (2.3a) and (b) (2.3b).

Case A[36]*:*

$Y(y)$ is a unimodal function that is continuous, strictly increasing on $[0, c)$, $Y(c)=1$, strictly decreasing on $(c, 1]$, and $Y(1)=0$ (2.3a)

Case B:

$Y(y)$ is a piecewise continuous function that is strictly increasing on each interval of $[0, c)$ and $(c, 1]$, $\lim_{y \to c^-} Y(y)=1$, and $Y(c)=0$ (2.3b)

Let $\mathbf{F}$ denote the mapping (2.1). The sufficient condition that $\mathbf{F}$ is invertible is $\mathbf{F}(S) \subset S$ and $\mathbf{F}(S_0) \cap \mathbf{F}(S_1) = \varnothing$. The generalized baker's transformation[37] is a special case of (2.1).

An orbit generated by the 1D map Y governs the basic properties of an orbit generated by the 2D map $\mathbf{F}$. Write $y_j \equiv Y^j(y_0)$ and $(x_j, y_j) \equiv \mathbf{F}^j(x_0, y_0)$. We have the following results.

Result 1. Assume that Y has a periodic orbit $\omega_n = \{y_0, y_1,..., y_{n-1}\}$ of period n. Then, $\mathbf{F}$ has a periodic orbit $\boldsymbol{\omega}_n$ of period n such that $\boldsymbol{\omega}_n = \{(x_0, y_0), (x_1, y_1),..., (x_{n-1}, y_{n-1})\}$. The map $\mathbf{F}$ has unique $\boldsymbol{\omega}_n$ if and only if Y has ω_n.

Result 2. Assume that Y has a stable periodic orbit ω_n. Then, the corresponding orbit of $\mathbf{F}$ to ω_n is stable, except for the case that $\boldsymbol{\omega}_n$ runs on $y = c$.

Result 3. Assume that Y has chaotic orbits on the attractor. $\mathbf{F}$ has a strange attractor and there are uncountably infinite chaotic orbits on the attractor which have the same y components and different x components. If and only if Y has a chaotic orbit does $\mathbf{F}$ have chaotic orbits.

$\Delta(y; a) \equiv [0, 1] \otimes [y, y+a]$ is a rectangle on S. We write the area of $\mathbf{F}(\Delta(y; a))$ as $\text{Area}\{\mathbf{F}(\Delta(y; a))\}$. Assume that the map Y has a chaotic attractor and a unique, absolutely continuous invariant measure, denoted by P_Y, on the attractor. Then, $\mathbf{F}$ has a strange attractor $\boldsymbol{\Omega}$ and a unique natural measure on $\boldsymbol{\Omega}$. We write the probability on $\boldsymbol{\Omega} \cap \mathbf{F}(\Delta(y; a))$ as $P_F[\mathbf{F}(\Delta(y; a))]$. We consider a family of maps which have different $t_0(y)$ and $t_1(y)$ and have the same κ_0, κ_1, and Y. Let $\mathscr{F}_t$ denote the family and contain the map $^*\mathbf{F} = (^*X, Y)$ such that

$$^*X(x, y) = \begin{cases} \kappa_0 x, & \text{for} \quad 0 \leqslant y < c \\ \kappa_1 x + 1 - \kappa_1, & \text{for} \quad c \leqslant y \leqslant 1 \end{cases} \tag{2.4}$$

Lemma 4. For $\mathbf{F}$ and $*\mathbf{F} \in \mathscr{F}_t$,

$$\text{Area}\{\mathbf{F}^j(\Delta(y; a))\} = \text{Area}\{*\mathbf{F}^j(\Delta(y; a))\} \qquad (j = 0, 1, 2,...) \tag{2.5}$$

$$\begin{aligned} P_F[\mathbf{F}^j(\Delta(y; a))] &= P_F^*[*\mathbf{F}^j(\Delta(y; a))] \\ &= P_Y[[y, y+a]] \qquad (j = 0, 1, 2,...) \end{aligned} \tag{2.6}$$

The y components of $\mathbf{F}^j(x', y')$ and $*\mathbf{F}^j(x', y')$ for $(x', y') \in \Delta(y; a)$ are $Y^j(y')$. If $\mathbf{F}^k(x', y') \in S(\sigma_k)$ for $0 \leqslant k \leqslant j-1$, then $*\mathbf{F}^k(x', y') \in S(\sigma_k)$, where $S(\sigma) \equiv S_\sigma$. Let $L\{[x', x'']; y'\}$ be a segment parallel to the x axis whose end points are (x', y') and (x'', y'). Write the length of L as $|L|$. Since

$$|\mathbf{F}(L\{[x', x'']; y'\})| = \kappa(\sigma)\, |x'' - x'| = |*\mathbf{F}(L\{[x', x'']; y'\})|$$

for $L\{[x', x'']; y'\} \subset S(\sigma)$, we have

$$\begin{aligned} |\mathbf{F}^j(L\{[x', x'']; y'\})| &= |x'' - x'| \prod_{k=0}^{j-1} \kappa(\sigma_k) \\ &= |*\mathbf{F}^j(L\{[x', x'']; y'\})| \qquad (j = 1, 2, 3,...) \end{aligned} \tag{2.7}$$

when $\mathbf{F}^k(x', y') \in S(\sigma_k)$ for $0 \leqslant k \leqslant j-1$, where we put $\kappa(\sigma) = \kappa_\sigma$. Therefore, we obtain (2.5). It is obvious that (2.6) holds, because $\mathbf{F}$ and $*\mathbf{F}$ conserve the probability.

Finally, we study local Lyapunov exponents around a reference orbit $\boldsymbol{\omega} = \{(x_0, y_0), (x_1, y_1), (x_2, y_2),...\}$ of $\mathbf{F}$. The local Lyapunov exponents are given by

$$\Lambda_j(\mathbf{x}_0; n) \equiv \frac{1}{n} \ln \left| \left(\frac{\partial \mathbf{x}_n}{\partial \mathbf{x}_0}\right) \mathbf{e}_j(\mathbf{x}_0; n) \right| \qquad (j = 1, 2) \tag{2.8}$$

where $(\partial \mathbf{x}_n / \partial \mathbf{x}_0)$ is the Jacobian matrix, $\mathbf{e}_j(\mathbf{x}_0; n)$ its normalized right-eigenvector, and $|\cdots|$ the Euclidean norm. I have assumed the existence of the Jacobian matrix for all $\mathbf{x}_0$ with respect to the probability measure $P(\mathbf{x}_0)$ of an initial value. Since a tangent map of $\mathbf{F}$ is a triangular matrix, the Jacobian matrix becomes

$$\frac{\partial \mathbf{x}_n}{\partial \mathbf{x}_0} = \begin{pmatrix} \prod_{k=0}^{n-1} X_{x;k} & \sum_{j=0}^{n-1} \left[\left(\prod_{k=j+1}^{n-1} X_{x;k} \right) X_{y;j} \prod_{i=0}^{j-1} Y'(y_i) \right] \\ 0 & \prod_{k=0}^{n-1} Y'(y_k) \end{pmatrix} \tag{2.9}$$

where we put $X_{z;k} \equiv (\partial x_{k+1}/\partial z_k)$ for $z = x$ and y and $Y'(y) = dY/dy$. Therefore, we get

$$\Lambda_1(\mathbf{x}_0; n, \mathbf{F}) = \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{dY(y_k)}{dy_k} \right| \tag{2.10a}$$

$$\Lambda_2(\mathbf{x}_0; n, \mathbf{F}) = \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{\partial X(x_k, y_k)}{\partial x_k} \right| \tag{2.10b}$$

When $\mathbf{x}_k$ falls into $S(\sigma_k)$ for each k $(0 \leqslant k \leqslant n-1)$, it turns out from (2.7) that

$$\Lambda_2(\mathbf{x}_0; n, \mathbf{F}) = \frac{1}{n} \sum_{k=0}^{n-1} \ln \kappa(\sigma_k) \tag{2.11}$$

Assume that $\boldsymbol{\omega}$ is an orbit of $\mathbf{F} \in \mathscr{F}_t$. Then any $\hat{\mathbf{F}} \in \mathscr{F}_t$ has an orbit, the y components of which are the same as those of $\boldsymbol{\omega}$. Since (2.10a) and (2.11) depend only on the y components of the reference orbit, the corresponding orbits of $\hat{\mathbf{F}}$ to $\boldsymbol{\omega}$ have the same local Lyapunov exponents as $\boldsymbol{\omega}$.

The right-eigenvectors of (2.9) for the eigenvalues Λ_1 and Λ_2 are

$$\mathbf{e}_1(\mathbf{x}_0; n, \mathbf{F}) = \begin{pmatrix} \sum_{j=0}^{n-1} \left\{ [X_{y;j}/Y'(y_j)] \prod_{k=j+1}^{n-1} [X_{x;k}/Y'(y_k)] \right\} \\ 1 - \prod_{j=0}^{n-1} [X_{x;j}/Y'(y_j)] \end{pmatrix} \tag{2.12}$$

and $\mathbf{e}_2(\mathbf{x}_0; n, \mathbf{F}) = (1, 0)^T$, respectively, where T means taking the column vector. I assume that the strange attractor contains a smooth line $x = g(y; x_n, y_n)$ passing through the point $\mathbf{x}_n \in \boldsymbol{\omega}$ in a neighborhood of $\mathbf{x}_n$.[13,33] The direction tangent to the attractor at $\mathbf{x}_n$ is given by the vector $\mathbf{u}_1(\mathbf{x}_n; \mathbf{F})$:

$$\mathbf{u}_1(\mathbf{x}_n; \mathbf{F}) = \left(\left(\frac{\partial g(y; x_n, y_n)}{\partial y} \right)_{y = y_n}, 1 \right)^T \tag{2.13}$$

Consider an orbit $\boldsymbol{\omega}' = \{..., (x'_0, y'_0), (x'_1, y'_1), ..., (x'_n, y'_n)\}$ which passes a point (x'_n, y'_n) on the line. Since $x'_{k+1} - x_{k+1} = X(x'_k, y'_k) - X(x_k, y_k)$ and $y'_{k+1} - y_{k+1} = Y(y'_k) - Y(y_k)$, one has

$$\left(\frac{\partial g(y; \mathbf{x}_n)}{\partial y} \right)_{y = y_n} = \sum_{j=-\infty}^{n-1} \frac{X_{y;j}}{Y'(y_j)} \prod_{k=j+1}^{n-1} \frac{X_{x;k}}{Y'(y_k)} \tag{2.14}$$

where I assumed convergence on the right-hand side of (2.14). For sufficiently large n, the direction of $\mathbf{e}_1(\mathbf{x}_0; n, \mathbf{F})$ in (2.12) coincides with

$\mathbf{u}_1(\mathbf{x}_n; \mathbf{F})$, which is a vector tangent to the attractor at the last point of the reference orbit.[14]

If $\boldsymbol{\omega}$ passes through a point where the derivative of Y vanishes, then we write $\Lambda_1 \equiv -\infty$. When the derivative of $t_0(y)$ or $t_1(y)$ becomes infinite at some point, the Jacobian matrix (2.9) is not defined if $\boldsymbol{\omega}$ passes through this point. In this case, $\boldsymbol{\Lambda}$ is still defined by (2.10) if possible. As long as the reference orbit passed the divergent point of the derivative of t_0 or t_1 in its past, (2.14) does not converge and the direction of $\mathbf{e}_1(\mathbf{x}_0; n, \mathbf{F})$ may be different from $\mathbf{u}_1(\mathbf{x}_n; \mathbf{F})$. Remark that the attractor is nonhyperbolic not only in the case where the attractor contains a vanishing point of the derivative of Y, but also in the case where the attractor contains a divergent point of the derivative of t_0 or t_1.

3. THERMODYNAMIC APPROACH

Assume that the map $\mathbf{F}$ has a strange attractor $\boldsymbol{\Omega}$. Then, the 1D map Y has a chaotic attractor and a unique, absolutely continuous invariant measure P_Y on its attractor. $\mathbf{F}$ has the invariant measure P_F on $\boldsymbol{\Omega}$, related to P_Y by (2.6). In this section, we construct the statistical thermodynamics of the scaling indices of the natural measure in the space of symbol sequence.[35,38] Without loss of generality, we take $^*\mathbf{F}$ given by (2.4) as $\mathbf{F}$.

3.1. Microscopic States of α and β

To study a probability measure on a strange attractor in terms of scaling indices, we first cover the attractor with a set of small boxes. However, it is ambiguous which boxes are chosen for the covering as well as how the boxes cover the attractor completely. We use a dynamical partition for the covering.[24,39] Divide the interval [0, 1] into two intervals

$$J(\sigma) \equiv \begin{cases} [0, c) & \text{for} \quad \sigma = 0 \\ [c, 1] & \text{for} \quad \sigma = 1 \end{cases} \tag{3.1}$$

Y is monotone and continuous on $J(\sigma)$. We can partition $J(0)$ and $J(1)$ into intervals on which Y^k is monotone and continuous:

$$J(\sigma_1 \sigma_2 \cdots \sigma_k) \equiv J(\sigma_1) \cap Y^{-1}(J(\sigma_2 \cdots \sigma_k)) \qquad \text{for} \quad k = 2, 3, \ldots \tag{3.2}$$

Partition the unit square S into rectangles

$$\Delta(\sigma_1 \sigma_2 \cdots \sigma_k) \equiv [0, 1] \otimes J(\sigma_1 \sigma_2 \cdots \sigma_k) \tag{3.3}$$

Each box is defined as

$$\Omega_{nm}(\sigma_1 \cdots \sigma_n; \sigma_{n+1} \cdots \sigma_{n+m}) \equiv \text{closure}\{\mathbf{F}^n(\Delta(\sigma_1 \sigma_2 \cdots \sigma_{n+m}))\} \tag{3.4}$$

The set of boxes Ω_{nm},

$$C(n, m) \equiv \{\Omega_{nm}(\sigma_1 \cdots \sigma_n; \sigma_{n+1} \cdots \sigma_{n+m});$$
$$\text{Area}\{\Omega_{nm}(\sigma_1 \cdots \sigma_n; \sigma_{n+1} \cdots \sigma_{n+m})\} \neq 0\} \qquad (3.5)$$

is capable of covering $\mathbf{\Omega}$ completely and consists of boxes of various sizes. $C(n, m)$, called the cover of natural size, is slightly different from the dynamical partition of the attractor[35] because the attractor may not run along the sides $x=0$ and $x=1$ of the unit square S.

For simplicity, we put $\boldsymbol{\sigma} = (\sigma_1 \sigma_2 \cdots \sigma_{n+m})$, $\boldsymbol{\sigma}_x = (\sigma_1 \sigma_2 \cdots \sigma_n)$, and $\boldsymbol{\sigma}_y = (\sigma_{n+1} \cdots \sigma_{n+m})$. The box $\Omega_{nm}(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$ is a rectangle which has two sides parallel to the x axis. The size of a box is characterized by the horizontal and vertical lengths of the sides of the box, $l_x(\boldsymbol{\sigma}_x)$ and $l_y(\boldsymbol{\sigma}_y; \boldsymbol{\sigma}_x)$, respectively; generally, the box is not a rectangle. Then, $l_y(\boldsymbol{\sigma}_y; \boldsymbol{\sigma}_x)$ is defined as the vertical distance between the sides parallel to the x axis, while $l_x(\boldsymbol{\sigma}_x)$ remains in the previous definition. If there exists a Markov partion of the attractor of Y, which is given by $\{J(\sigma_1 \sigma_2 \cdots \sigma_M)\}$, then it turns out that

$$l_y(\boldsymbol{\sigma}_y; \boldsymbol{\sigma}_x) = l_y(\boldsymbol{\sigma}_y) = |J(\sigma_{n+1} \cdots \sigma_{n+m})| \qquad \text{for} \quad n, m \geqslant M \qquad (3.6)$$

A Markov partition of an attractor of Y is a set of intervals all endpoints of which are mapped on the endpoints by Y.[40] In the following, I assume the existence of a Markov partition. As n and m increase, the sizes of the boxes decrease exponentially fast. These exponential dampings are characterized by the size indices of the box, defined on each box as

$$\beta_x(\boldsymbol{\sigma}_x) \equiv -\frac{1}{n} \ln l_x(\boldsymbol{\sigma}_x), \qquad \beta_y(\boldsymbol{\sigma}_y) \equiv -\frac{1}{m} \ln l_y(\boldsymbol{\sigma}_y) \qquad (3.7)$$

The natural measure on the attractor of $\mathbf{F}$ is described in terms of the natural measure of Y from Lemma 4. When the probability on $\mathbf{\Omega}$ covered with $\Omega_{nm}(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$ is denoted by $P_F(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$ and the probability of Y on $J(\boldsymbol{\sigma})$ by $P_Y(\boldsymbol{\sigma})$, (2.6) becomes

$$P(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y) \equiv P_F(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y) = P_Y(\boldsymbol{\sigma}) \qquad (3.8)$$

Scaling indices $\boldsymbol{\alpha} = (\alpha_x, \alpha_y)$ of the natural measure of $\mathbf{F}$ are defined on each box by

$$-n\alpha_x(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)\, \beta_x(\boldsymbol{\sigma}_x) - m\alpha_y(\boldsymbol{\sigma}_y; \boldsymbol{\sigma}_x)\, \beta_y(\boldsymbol{\sigma}_y) \equiv \ln P(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y) \qquad (3.9)$$

which measure the singularity of the probability $P(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$ with respect to the Lebesgue measure of the box.

3.2. Free Energy Functions

I introduce a new probability measure ρ on the cover $C(n; m)$, so that the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is constructed on the cover[18]:

$$\rho(\boldsymbol{\sigma}; q, \boldsymbol{\tau}) \equiv [1/\Xi_{nm}(q, \boldsymbol{\tau})] \exp[-qU(\boldsymbol{\sigma}; \boldsymbol{\tau}, n, m)] \tag{3.10}$$

where $\boldsymbol{\tau} = (\tau_x, \tau_y)$, the interaction $U(\boldsymbol{\sigma})$ is given by

$$\begin{aligned} qU(\boldsymbol{\sigma}) \equiv\ & n[q\alpha_x(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)\, \beta_x(\boldsymbol{\sigma}_x) - \tau_x \beta_x(\boldsymbol{\sigma}_x)] \\ & + m[q\alpha_y(\boldsymbol{\sigma}_y; \boldsymbol{\sigma}_x)\, \beta_y(\boldsymbol{\sigma}_y) - \tau_y \beta_y(\boldsymbol{\sigma}_y)] \end{aligned} \tag{3.11}$$

and the normalization factor $\Xi_{nm}(q, \boldsymbol{\tau})$ is

$$\Xi_{nm}(q, \boldsymbol{\tau}) \equiv \sum_{\{\sigma_1 \cdots \sigma_{n+m}\}}' \exp[-qU(\boldsymbol{\sigma}; \boldsymbol{\tau}, n, m)] \tag{3.12}$$

The summation in (3.12) must be taken over all elements of $C(n, m)$.

The statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be treated in a one-dimensional lattice system. Consider the space of symbol sequence, $\Theta \equiv \{0, 1\}^{Z'}$, which is a semi-infinite product space of $\{0, 1\}$; Z' is the set of all positive integers. Θ_K is the subspace of Θ that consists of every cylinder set of length K which starts from the first lattice point. Ω_{nm} in (3.4) gives a mapping of $C(n, m)$ to Θ_K, $K = n + m$. As Θ_K^* is the image of $C(n, m)$, the mapping given by (3.4) is a bijection from $C(n, m)$ to Θ_K^*; then, an element of Θ_K^* is a cylinder set of length K. We may consider the space Θ_K as the configuration space of Ising spins on a one-dimensional lattice of size K. The scaling indices $\boldsymbol{\alpha}$ and the size indices $\boldsymbol{\beta}$ of a box are quantities defined on Θ_K^*, such as a specific magnetization in the spin system. $U(\boldsymbol{\sigma})$ corresponds to the Hamiltonian of spin, q to the inverse temperature, and $\boldsymbol{\tau}$ to the external forces conjugate to $\boldsymbol{\beta}$.[10,38] However, the interactions among "spins" are generally inhomogeneous and anisotropic. And there is no translational invariance; we will see examples in Section 4 and 5. In the language of statistical thermodynamics,[41] we call $\rho(\boldsymbol{\sigma}; q, \boldsymbol{\tau})$ the finite-volume Gibbs states of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and $\Xi_{nm}(q, \boldsymbol{\tau})$ the partition function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Note that (3.10) is not an invariant measure of $\mathbf{F}$ such as discussed in refs. 40.

It is obvious from (3.12) that $\Xi_{nm}(q, \boldsymbol{\tau})$ is concave, monotone decreasing for q, and monotone increasing for τ_v ($v = x$ and y). Define the free energy function of α_v and β_v ($v = x$ and y) as

$$G_x(q, \tau_x; \tau_y) \equiv -\lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{n} \ln \Xi_{nm}(q, \tau_x, \tau_y) \tag{3.13a}$$

$$G_y(q, \tau_y; \tau_x) \equiv -\lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \ln \Xi_{nm}(q, \tau_x, \tau_y) \tag{3.13b}$$

assuming the existence of the thermodynamic limits. As the attractor of Y has the Markov partition, we have

$$G_x(q, \tau_x; \tau_y) = G_x(q, \tau_x), \qquad G_y(q, \tau_y; \tau_x) = G_y(q, \tau_y) \tag{3.14}$$

$G_v(q, \tau_v)$ is monotone and convex,[41,42] increases with respect to q, and decreases with respect to τ_v.

The definition of $\boldsymbol{\alpha}$ in (3.9) is insufficient for n and m fixed, because α_x and α_y cannot be uniquely determined on each box. Redefine α_x and α_y as

$$\alpha_x(\boldsymbol{\sigma}_x)\, \beta_x(\boldsymbol{\sigma}_x) \equiv -\frac{1}{n} \ln P_F(\boldsymbol{\sigma}_x) \tag{3.15}$$

$$\alpha_y(\boldsymbol{\sigma}_y; \boldsymbol{\sigma}_x)\, \beta_y(\boldsymbol{\sigma}_y) \equiv -\frac{1}{m} \ln \frac{P(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)}{P_F(\boldsymbol{\sigma}_x)} \tag{3.16}$$

for n and m fixed, where $P_F(\boldsymbol{\sigma}_x)$ is the probability on $\Omega_n(\boldsymbol{\sigma}_x)$ given by putting $m = 0$ in (3.4). $P_Y(\boldsymbol{\sigma})$ is written as

$$P_Y(\boldsymbol{\sigma}) = P_Y(\sigma_1 \cdots \sigma_M) \prod_{j=1}^{n+m-M} Q(\sigma_{j+1} \cdots \sigma_{j+M}; \sigma_j \cdots \sigma_{j+M-1}) \tag{3.17}$$

where the transition probability is given by

$$Q(\sigma_2 \cdots \sigma_{M+1}; \sigma_1 \cdots \sigma_M) \equiv \frac{P_Y(\sigma_1 \cdots \sigma_M \sigma_{M+1})}{P_Y(\sigma_1 \cdots \sigma_M)} \tag{3.18}$$

For $m \gg M$, (3.16) becomes

$$\alpha_y(\boldsymbol{\sigma}_y; \boldsymbol{\sigma}_x)\, \beta_y(\boldsymbol{\sigma}_y) \approx -\frac{1}{m} \ln \sum_{\{\sigma_x\}}{}' P(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y) = \alpha_y(\boldsymbol{\sigma}_y)\, \beta_y(\boldsymbol{\sigma}_y) \tag{3.19}$$

Therefore, the probability that the scaling indices and the size indices of a box take values in $[\boldsymbol{\alpha}, \boldsymbol{\alpha} + d\boldsymbol{\alpha}]$ and $[\boldsymbol{\beta}, \boldsymbol{\beta} + d\boldsymbol{\beta}]$ is factorized as

$$W(\boldsymbol{\alpha}, \boldsymbol{\beta}; n, m)\, d\boldsymbol{\alpha}\, d\boldsymbol{\beta} \approx \prod_{v = x, y} W_v(\alpha_v, \beta_v; n_v)\, d\alpha_v\, d\beta_v \qquad \text{for} \quad m \gg M$$

where $n_x = n$ and $n_y = m$. We assume the existence of the limits

$$s_{\alpha\beta; x}(\alpha_x, \beta_x) \equiv \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{n} \ln W(\boldsymbol{\alpha}, \boldsymbol{\beta}; n, m) \tag{3.20a}$$

$$s_{\alpha\beta; y}(\alpha_y, \beta_y) \equiv \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \ln W(\boldsymbol{\alpha}, \boldsymbol{\beta}; n, m) \tag{3.20b}$$

The large-deviation theory has proven the existence of these limits for some cases.[41] We call $s_{\alpha\beta;\nu}$ the entropy function of α_ν and β_ν. Having the limits in (3.20), we also have the limits in (3.13) and can write in the partial forms

$$G_\nu(q, \tau_\nu) = \min_{\alpha_\nu, \beta_\nu} \{(q-1)\,\alpha_\nu\beta_\nu - \tau_\nu\beta_\nu - s_{\alpha\beta;\nu}(\alpha_\nu, \beta_\nu)\} \tag{3.21}$$

for $\nu = x$ and y. The thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is obtained from $G_\nu(q, \tau_\nu)$. The averages of random variables such as α_ν and β_ν over the Gibbs ensemble (3.10) are given by

$$\langle\alpha_\nu\rangle = \sum_{\{\boldsymbol{\sigma}\}}{}' \alpha_\nu(\sigma_\nu)\,\rho(\boldsymbol{\sigma}; q, \boldsymbol{\tau}), \qquad \langle\beta_\nu\rangle = \sum_{\{\boldsymbol{\sigma}\}}{}' \beta_\nu(\boldsymbol{\sigma}_\nu)\,\rho(\boldsymbol{\sigma}; q, \boldsymbol{\tau}) \tag{3.22}$$

3.3. Generalized Partial Dimensions

$G_\nu(q, \tau_\nu)$ is continuous, monotone increasing for q, and monotone decreasing for τ_ν. Since $G_\nu(q, \tau_\nu)$ tends to $+\infty$ for $\tau_\nu \to -\infty$ and to $-\infty$ for $\tau_\nu \to +\infty$ (Fig. 2)

$$G_\nu(q, \tau_\nu) = 0 \tag{3.23}$$

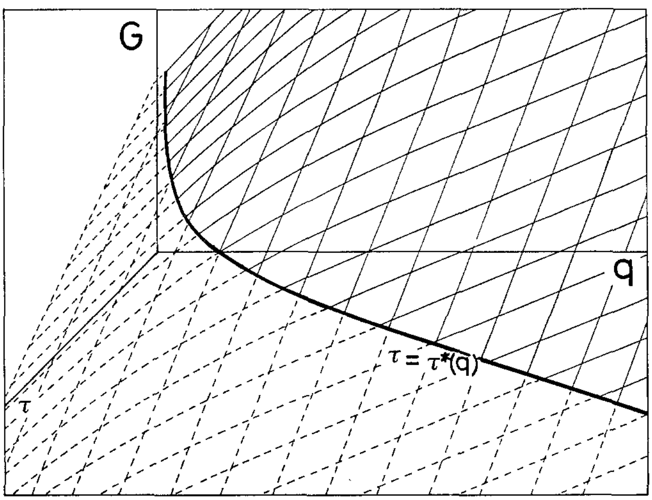


Fig. 2. Free energy function $G_\nu(q, \tau_\nu)$ of α_ν and β_ν. The function $G_\nu(q, \tau_\nu)$ is continuous, monotone increasing for q, and monotone decreasing for τ_ν. $G_\nu(q, \tau_\nu) = 0$ has a unique solution $\tau_\nu = \tau_\nu^*(q)$ which is a continuous and monotone-increasing function of q. Since $G_\nu(q, \tau_\nu)$ is convex, $\tau_\nu^*(q)$ is convex. The figure displays $G_x(q, \tau_x)$ of (4.5a) with $\lambda_0 = 3.0$, $\lambda_1 = 1.5$, $\kappa_0 = 0.35$, and $\kappa_1 = 0.2$. The heavy line denotes $\tau_x = \tau_x^*(q)$.

has a unique solution $\tau_v = \tau_v^*$, where τ_v^* is a continuous and monotone-increasing function of q. The function $\tau_v^*(q)$ is convex because $G_v(q, \tau_v)$ is convex. The generalized partial dimensions $D_v(q)$ are defined by

$$D_v(q) \equiv \frac{1}{q-1} \tau_v^*(q) \tag{3.24}$$

The generalized dimension $D(q)$ is the sum of $D_x(q)$ and $D_y(q)$:

$$D(q) = D_x(q) + D_y(q) \tag{3.25}$$

The attractor of Y is equal to the image of the projection of the attractor of $\mathbf{F}$ on the y axis. It is easy to construct the statistical thermodynamics of the scaling index and the size index of a box on the attractor of Y with respect to the cover $\{J(\sigma_1 \cdots \sigma_m);\ |J(\sigma_1 \cdots \sigma_m)| \neq 0\}$. The scaling index α and the size index β of a box and a finite-volume Gibbs state $\rho(\boldsymbol{\sigma}_1; q, \tau)$ are defined on each interval $J(\boldsymbol{\sigma}_1)$, $\boldsymbol{\sigma}_1 = (\sigma_1 \cdots \sigma_m)$, by

$$\alpha(\boldsymbol{\sigma}_1) \equiv \frac{\ln P_Y(\boldsymbol{\sigma}_1)}{\ln |J(\boldsymbol{\sigma}_1)|}, \qquad \beta(\boldsymbol{\sigma}_1) \equiv -\frac{1}{m} \ln |J(\boldsymbol{\sigma}_1)|$$

$$\rho(\boldsymbol{\sigma}_1; q, \tau) \equiv \frac{1}{\Xi_m(q, \tau)} \exp\{-m\beta(\boldsymbol{\sigma}_1)[q\alpha(\boldsymbol{\sigma}_1) - \tau]\}$$

where the normalization factor $\Xi_m(q, \tau)$ is the partition function of α and β. Then, it is obvious that the statistical thermodynamics of α_y and β_y is the same as that of α and β. Generally, there exists some ambiguity in the definition of partial dimensions. For a chaotic attractor of $\mathbf{F}$ the direction of $\mathbf{u}_1$ in (2.13) may be different from the y direction. If the attractor does not have a point where $\mathbf{u}_1$ is parallel to the x direction, $D_y(q)$ is equal to $D_1(q)$. However, the difference of $D_y(q)$ and $D_1(q)$ is essential in the other case. Remark that $D_1(q)$ has usually been used in the literature.[12,14,32]

3.4. Generalized Entropies

$P_Y(\sigma_1 \sigma_2 \cdots \sigma_n)$ gives the probability that a trajectory passes through the boxes $S(0)$ and $S(1)$ in the phase space in the order of $S(\sigma_1)$, $S(\sigma_2)$,..., and $S(\sigma_n)$. Since a time damping rate of the joint probability is defined by

$$\gamma(\sigma_1 \sigma_2 \cdots \sigma_n) \equiv -\frac{1}{n} \ln P_Y(\sigma_1 \sigma_2 \cdots \sigma_n) \tag{3.26}$$

it turns out that $\gamma(\sigma_1 \cdots \sigma_n) = \alpha_x(\sigma_1 \cdots \sigma_n)\, \beta_x(\sigma_1 \cdots \sigma_n)$. The free energy function of γ,

$$\Psi(q) \equiv -\lim_{n \to \infty} \frac{1}{n} \ln \sum_{\{\sigma_1 \cdots \sigma_n\}}{}' [P_Y(\sigma_1 \cdots \sigma_n)]^q \tag{3.27}$$

is related to the free energy function of α_x and β_x by

$$\Psi(q) = G_x(q, \tau = 0) \tag{3.28}$$

so that we have

$$\Psi(q) = \min_{\gamma} \{(q-1)\gamma - \max_{\alpha_x} [s_{\alpha\beta;x}(\alpha_x, \gamma/\alpha_x)]\} \tag{3.29}$$

$h(\gamma)$ and $\beta_x f_x(\alpha_x, \beta_x)$ are the Legendre–Fenchel transforms[41] of $\Psi(q)$ and $G_x(q, \tau_x)$, respectively. Then, (3.29) leads to the relation

$$h(\gamma) = \max_{\alpha_x} \{(\gamma/\alpha_x) f_x(\alpha_x, \gamma/\alpha_x)\} = (\gamma/\alpha_x^*) f_x(\alpha_x^*, \gamma/\alpha_x^*) \tag{3.30}$$

where α_x^* is the value of α_x giving the maximum of (3.30). We sometimes have examples where $\alpha_x(\boldsymbol{\sigma}_x)$ depends explicitly on $\beta_x(\boldsymbol{\sigma}_x)$. If α_x is related to β_x in a unique way, α_x and γ have a one-to-one correspondence. Then, we have

$$h(\gamma)/\gamma = f_x(\alpha_x(\gamma))/\alpha_x(\gamma) \tag{3.31}$$

This relation holds when the random variables γ and α are completely dependent.[15,16] In general, γ and α have an independent part, so that (3.31) does not hold and must be rewritten as (3.30).

3.5. Theorem

In this section, we have studied the attractor of the map (2.4). As long as the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is defined on the cover (3.5) by (3.7), (3.10), (3.11), (3.15), and (3.16), all results still hold for all chaotic attractors of maps belonging to the same $\mathscr{F}_t$. We conclude with the following result.

Theorem 5. For $\mathbf{F}$, $\hat{\mathbf{F}} \in \mathscr{F}_t$, the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on a chaotic attractor of $\mathbf{F}$ with respect to the cover of natural size is equal to that on a chaotic attractor of $\hat{\mathbf{F}}$ with respect to the cover of natural size.

From the definition of the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and Lemma 4, it is obvious that the theorem holds. Note that this theorem still holds even if there exists no Markov partition of the attractor of Y.

4. EXAMPLES: HYPERBOLIC ATTRACTORS

In this section, I give two examples which are attractors of piecewise linear maps and analyze the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ concretely. The attractors consist only of hyperbolic points. The statistical thermodynamics of local Lyapunov exponents is also studied in relation (1.1).

4.1. Example 1

Consider (see Fig. 3)

$$\begin{aligned} X(x, y) &= \kappa_0 x, & Y(y) &= \lambda_0 y, & &\text{if } 0 \leqslant y < c \\ X(x, y) &= \kappa_1 x + 1 - \kappa_1, & Y(y) &= \lambda_1 (1 - y), & &\text{if } c \leqslant y \leqslant 1 \end{aligned} \tag{4.1}$$

where $\lambda_0 = 1/c$ and $\lambda_1 = 1/(1-c)$. The mapping (4.1) has the same fractal structures of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as the attractor of the generalized baker's transforma-

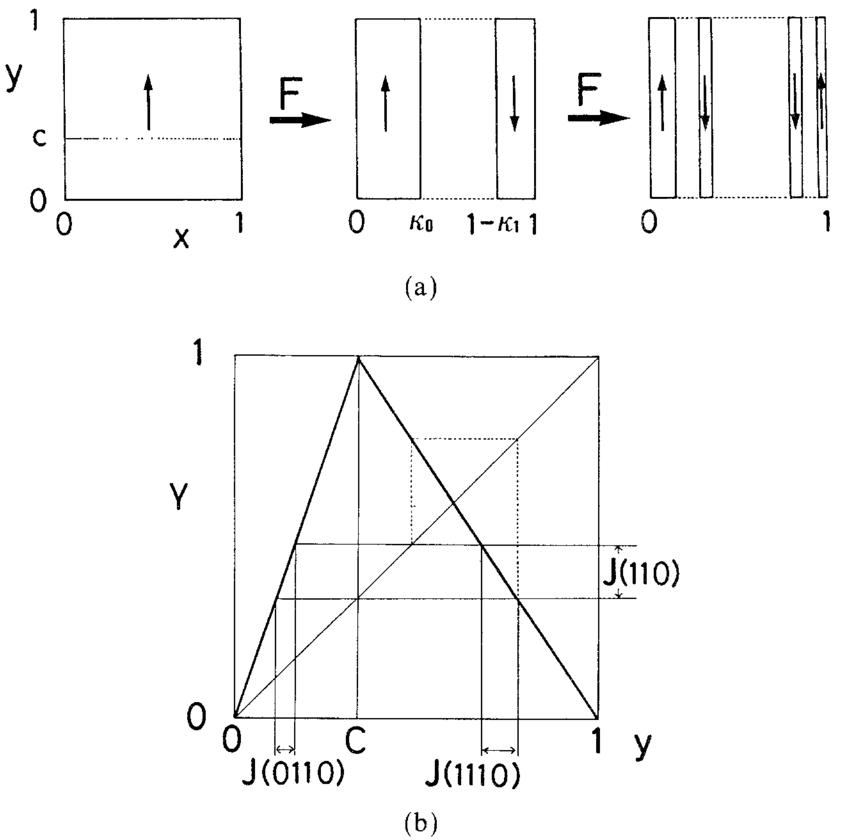


Fig. 3. (a) Schematic illustrating the mapping (4.1). (b) the one-dimensional map Y in (4.1). On using Y, the interval $[0, 1]$ is partitioned into small intervals. Both intervals of $J(0110)$ and $J(1110)$ are mapped onto $J(110)$ under the mapping.

tion. This example is therefore instructive. Following Section 3.1, we obtain $C(n, m)$, which is isomorphic to Θ_K, $K = n + m$. The length of the interval $J(\sigma_1 \cdots \sigma_K)$ is

$$|J(\sigma_1 \cdots \sigma_K)| = \prod_{j=1}^{K} \{1/\lambda(\sigma_j)\} \tag{4.2}$$

where $\lambda(\sigma) = \sigma\lambda_1 + (1-\sigma)\lambda_0$ for $\sigma = 0, 1$. Since the natural measure of Y is the length of an interval, (3.8) becomes

$$P(\sigma_1 \cdots \sigma_n; \sigma_{n+1} \cdots \sigma_{n+m}) = |J(\sigma_1 \cdots \sigma_{n+m})| \tag{4.3}$$

The attractor of $\mathbf{F}$ has a uniform scaling index in the y direction. $\alpha_x(\boldsymbol{\sigma}_x)$ is dependent explicitly on $\beta_x(\boldsymbol{\sigma}_x)$:

$$\begin{aligned} &\alpha_x(\boldsymbol{\sigma}_x)\,\beta_x(\boldsymbol{\sigma}_x)\ln(\kappa_0/\kappa_1) + \beta_x(\boldsymbol{\sigma}_x)\ln(\lambda_0/\lambda_1) \\ &\quad = (\ln\lambda_1)\ln\kappa_0 - (\ln\lambda_0)\ln\kappa_1 \equiv A \end{aligned} \tag{4.4}$$

The free energy function of α_v and β_v becomes

$$G_x(q, \tau) = -\ln[(1/\lambda_0)^q (1/\kappa_0)^\tau + (1/\lambda_1)^q (1/\kappa_1)^\tau] \tag{4.5a}$$

$$G_y(q, \tau) = -\ln(\lambda_0^{\tau-q} + \lambda_1^{\tau-q}) \tag{4.5b}$$

Applying (3.23) to (4.5) yields

$$\frac{1}{\lambda_0^q}\exp(-\tau_x^* \ln\kappa_0) + \frac{1}{\lambda_1^q}\exp(-\tau_x^* \ln\kappa_1) = 1 \tag{4.6a}$$

$$\tau_y^* = q - 1 \tag{4.6b}$$

Since $G_x(q, \tau)$ is differentiable, it is easy to obtain the Legendre transform of $G_x(q, \tau)$. Using (4.4), we have the explicit form of $f_x(\alpha_x)$, which is equal to the Legendre transform of $\tau_x^*(q)$. Relations (4.6a) and (4.6b) are the same as those obtained for the generalized baker's transformation.[3,5] Since (4.4) relates α_x and β_x, (3.31), i.e., the relation between the spectra of α and γ, holds.[16]

Local Lyapunov exponents are given by (2.10). The local Lyapunov exponents determined during a time interval m have the same values for all reference orbits whose initial values are on the part of the attractor included in the box $\Omega_{nm}(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$: For any $\mathbf{x}_0 \in \boldsymbol{\Omega} \cap \Omega_{nm}(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$,

$$\Lambda_1(\mathbf{x}_0; m) = \Lambda_1(\boldsymbol{\sigma}_y; m) \equiv \frac{1}{m}\sum_{j=1}^{m} \ln\lambda(\sigma_{j+n})$$

$$\Lambda_2(\mathbf{x}_0; m) = \Lambda_2(\boldsymbol{\sigma}_y; m) \equiv \frac{1}{m}\sum_{j=1}^{m} \ln\kappa(\sigma_{j+n})$$

Note that $\Lambda_1(\boldsymbol{\sigma}_y; m)$ is equal to the size index β_y of the box. A relation between Λ_1 and Λ_2 similar to (4.4) holds:

$$\Lambda_1(\boldsymbol{\sigma}_y; m)\ln(\kappa_0/\kappa_1) - \Lambda_2(\boldsymbol{\sigma}_y; m)\ln(\lambda_0/\lambda_1) = \Lambda \tag{4.7}$$

The free energy function of $\boldsymbol{\Lambda}$ is defined by

$$\Phi(z_1, z_2) \equiv -\lim_{n\to\infty}\frac{1}{n}\ln\left\{\int dP(\mathbf{x}_0)\exp\left[-n\sum_{j=1}^{2} z_j\Lambda_j(\mathbf{x}_0; n)\right]\right\} \tag{4.8}$$

As (4.3) gives the probability of the initial value, it turns out that

$$\Phi(z_1, z_2) = -\ln[(1/\lambda_0)^{z_1+1}(1/\kappa_0)^{z_2} + (1/\lambda_1)^{z_1+1}(1/\kappa_1)^{z_2}] \tag{4.9}$$

Comparing (4.5) and (4.9), we can write

$$G_x(q, \tau) = \Phi(q-1, \tau) \tag{4.10}$$

Using (4.6b) and the definition of τ_x^*, we get

$$\Phi(\tau_y^*, \tau_x^*) = 0 \tag{4.11}$$

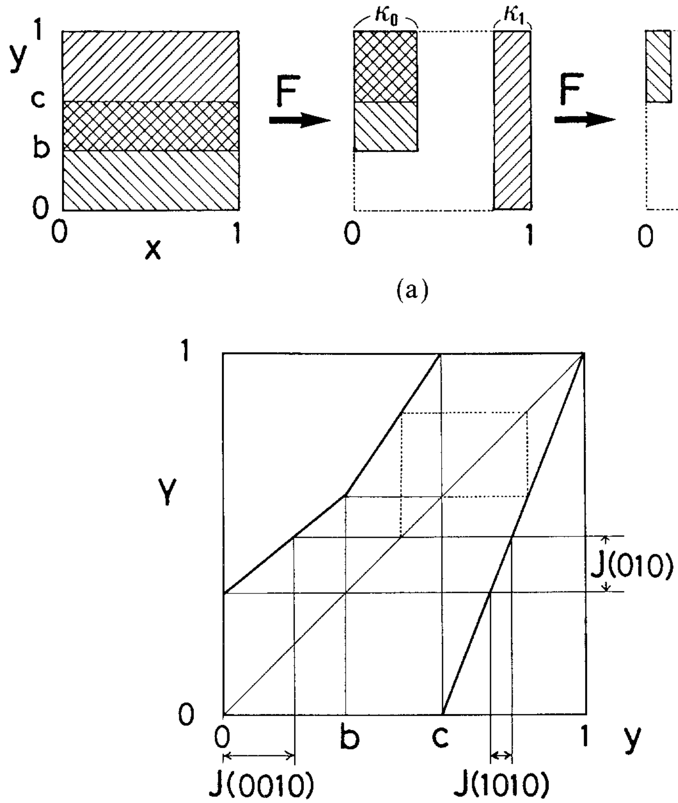
This corresponds to the relation (1.1) obtained by Morita *et al.*[14] Note that (1.1) and (4.11) are not equivalent. $\tau_1(q)$ and $\tau_2(q)$ may be different from $\tau_y^*(q)$ and $\tau_x^*(q)$, respectively, because $\mathbf{u}_1$ and $\mathbf{u}_2$ are not always parallel to the y and x axes, respectively. Theorem 5 says that the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is invariant for the attractors of the maps belonging to the same family $\mathscr{F}_t$. Therefore, (4.11) may hold on the non-hyperbolic attractors containing homoclinic tangency points, where the difference of α_1 and α_y seems to be crucial. It is obvious from (3.28) and (4.10) that (1.2) holds.

4.2. Example 2

Consider (see Fig. 4)

$$\begin{aligned}
&X(x, y) = \kappa_0 x, && Y(y) = \lambda_a y + b, && \text{if } 0 \leqslant y < b\\
&X(x, y) = \kappa_0 x, && Y(y) = \lambda_b(y-b) + c, && \text{if } b \leqslant y < c\\
&X(x, y) = \kappa_1(x-1) + 1, && Y(y) = \lambda_c(y-c), && \text{if } c \leqslant y \leqslant 1
\end{aligned} \tag{4.12}$$

where $\lambda_a = (c-b)/b$, $\lambda_b = (1-c)/(c-b)$, and $\lambda_c = 1/(1-c)$. The $C(n, m)$ of (4.12) does not contain the boxes labeled by such symbol sequences as $\sigma_{j-1} = \sigma_j = \sigma_{j+1} = 0$ for $j = 2, 3,..., n+m-1$. The subspace Θ_K^* of symbol sequence, isomorphic to $C(n, m)$, is



(a)

(b)

Fig. 4. (a) Schematic illustrating the mapping (4.12). (b) The one-dimensional map Y in (4.12).

$$\Theta_K^* = \{(\sigma_1\sigma_2\cdots\sigma_K);\ \sigma_j \in \{0,1\} \text{ and } \sigma_{j-1}^2+\sigma_j^2+\sigma_{j+1}^2 \neq 0 \text{ for all } j\}$$

where $K = n + m$. The length of the interval (3.2) is given by

$$|J(\sigma_1\cdots\sigma_K)| = |J(\sigma_{K-1}\sigma_K)| \prod_{j=1}^{K-2} \frac{1}{\lambda(\sigma_j\sigma_{j+1})} \quad \text{for } (\sigma_1\cdots\sigma_K) \in \Theta_K^* \tag{4.13}$$

where $\lambda(\sigma\sigma') = (1-\sigma)(1-\sigma')\,\lambda_a + (1-\sigma)\,\sigma'\lambda_b + \sigma\lambda_c$ for $\sigma, \sigma' \in \{0, 1\}$. The probability measure is given by

$$P_Y(\sigma_1\cdots\sigma_K) = |J(\sigma_1\cdots\sigma_K)|\,\frac{P_Y(\sigma_{K-1}\sigma_K)}{|J(\sigma_{K-1}\sigma_K)|} \quad \text{for } (\sigma_1\cdots\sigma_K) \in \Theta_K^* \tag{4.14}$$

The interaction can now be written explicitly as

$$qU(\boldsymbol{\sigma}) = \sum_{j=1}^{n} \{q[(1-\sigma_j)(1-\sigma_{j+1}) \ln \lambda_a + (1-\sigma_j)\,\sigma_{j+1} \ln \lambda_b + \sigma_j \ln \lambda_c] + \tau_x[(1-\sigma_j) \ln \kappa_0 + \sigma_j \ln \kappa_1]\} + (q-\tau_y) \sum_{j=1}^{m} \{(1-\sigma_{j+n})(1-\sigma_{j+1+n}) \ln \lambda_a + (1-\sigma_{j+n})\,\sigma_{j+1+n} \ln \lambda_b + \sigma_{j+n} \ln \lambda_c\} + O(1) \quad \text{for } \boldsymbol{\sigma} \in \Theta_K^* \tag{4.15}$$

The interaction is like the Hamiltonian of a two-component (i.e., A and B atoms) alloy in a one-dimensional lattice system: It is possible only for a B atom to occupy a lattice point at the center when A atoms occupy the lattice points at both sides. There are interactions between the nearest-neighbor atoms. The strength of the interactions as well as the chemical potential of each atom is different between the first n lattice points and the others. Thus, the system has no translational invariance.

By using the transfer matrices

$$\mathbf{M}_x(q, \tau) \equiv \begin{pmatrix} 0 & \lambda_a^{-q}\kappa_0^{-\tau} & 0 \\ 0 & 0 & \lambda_b^{-q}\kappa_0^{-\tau} \\ \lambda_c^{-q}\kappa_1^{-\tau} & \lambda_c^{-q}\kappa_1^{-\tau} & \lambda_c^{-q}\kappa_1^{-\tau} \end{pmatrix} \tag{4.16a}$$

$$\mathbf{M}_y(q-\tau) \equiv \mathbf{M}_x(q-\tau, 0) \tag{4.16b}$$

we can write the partition function as

$$\Xi_{nm}(q, \tau_x, \tau_y) = \mathbf{a}_1 \mathbf{M}_x^n(q, \tau_x)\, \mathbf{M}_y^{m-2}(q-\tau_y, 0)\, \mathbf{a}_2 \tag{4.17}$$

where $\mathbf{a}_1 = (1, 1, 1)$ and $\mathbf{a}_2$ is a three-dimensional column vector with positive elements. The maximum eigenvalue of $\mathbf{M}_v$ gives the free energy function of α_v and β_v ($v = x$ and y):

$$0 = \exp[-3G_x(q, \tau)] - \lambda_c^{-q}\kappa_1^{-\tau} \exp[-2G_x(q, \tau)] - (\lambda_b\lambda_c)^{-q}(\kappa_0\kappa_1)^{-\tau} \exp[-G_x(q, \tau)] - (\lambda_a\lambda_b\lambda_c)^{-q}(\kappa_0^2\kappa_1)^{-\tau} \tag{4.18a}$$

$$G_y(q, \tau) = G_x(q-\tau, 0) \tag{4.18b}$$

By their definition, τ_x^* and τ_y^* satisfy

$$\lambda_c^{-q} \exp(-\tau_x^* \ln \kappa_1) + (\lambda_b \lambda_c)^{-q} \exp[-\tau_x^* \ln(\kappa_0 \kappa_1)] + (\lambda_a \lambda_b \lambda_c)^{-q} \exp[-\tau_x^* \ln(\kappa_0^2 \kappa_1)] = 1 \tag{4.19a}$$

$$\tau_y^* = q - 1 \tag{4.19b}$$

The free energy function of $\mathbf{\Lambda}$ is written as

$$\Phi(z_1, z_2) = -\lim_{m \to \infty} \frac{1}{m-1} \ln \left\{ \sum_{\{\sigma\}}' \prod_{j=1}^{m-1} \left[\left(\frac{1}{\lambda(\sigma_j \sigma_{j+1})} \right)^{z_1+1} \left(\frac{1}{\kappa(\sigma_j)} \right)^{z_2} \right] \right\} \tag{4.20}$$

Using $\mathbf{M}_x(z_1 + 1, z_2)$ in (4.16a), we get

$$\Phi(z_1, z_2) = G_x(z_1 + 1, z_2) \tag{4.21}$$

Thus, we have the relation in (4.9) again. From (4.19), it turns out that (1.1) and (1.2) hold.

We have used the symbol space of $\{0, 1\}$ to study the strange attractors of $\mathbf{F}$. Other symbol spaces may be used for simplicity. For example, the analysis in this subsection can be easily done by using the new symbol space of $\{-1, 0, 1\}$, each element of which corresponds to each interval of the minimum Markov partition of the attractor of Y. For the attractor of $\mathbf{F}$ whose Y is given in Fig. 5, the analysis of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on $C(n, m)$ of (3.4)

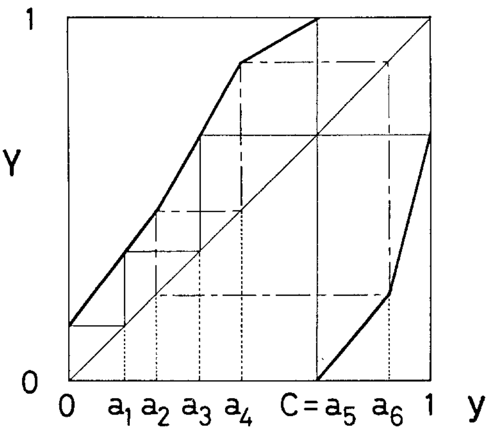


Fig. 5. A one-dimensional map Y in $\mathbf{F}$. The parameters a_j must satisfy suitable relations. The attractor of Y then has Markov partitions $\{[a_{j-1}, a_j)$ for $1 \leqslant j \leqslant 6$ and $[a_6, 1]\}$ and $\{[0, a_1), [a_1, a_3), [a_3, a_5), [a_5, 1]\}$.

is difficult because $\boldsymbol{\alpha}(\boldsymbol{\sigma})$ and $\boldsymbol{\beta}(\boldsymbol{\sigma})$ yield long-range interactions between symbols at different lattice points. In general, let us assume a piecewise linear map of Y has a proper Markov partition with k elements such that the slope of Y is constant on each of its elements. We also assume Y is mixing. Generate a dynamical partition of Y from the proper Markov partition. And construct a cover, of natural size, of the attractor of $\mathbf{F}$. Then, the free energy functions of α_v and β_v and of Λ are written in terms of the logarithm of the maximum eigenvalues of $k \times k$ irreducible nonnegative matrices. Each element of the $k \times k$ matrices is an analytic function of (q, τ_v) or (z_1, z_2). From the Frobenius theorem of a $k \times k$ irreducible nonnegative matrix,[43] it turns out that the free energy functions are analytic. Therefore, there is no phase transition in the thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and of Λ on the attractor of a piecewise linear $\mathbf{F}$ with the proper Markov partition of Y. Of course, in the limit $k \to \infty$, a phase transition may occur in the thermodynamics. The result obtained by Mori *et al.*[20] holds for just this case.

5. A EXAMPLE: A NONHYPERBOLIC ATTRACTOR

Consider (5.1) as an example of a nonhyperbolic chaotic system (see Fig. 6)

$$\begin{aligned} &X(x, y) = \kappa_0 x, && Y(y) = 4y(1-y), && \text{for } 0 \leqslant y < c = 1/2 \\ &X(x, y) = \kappa_1(x-1)+1, && Y(y) = (2y-1)^2, && \text{for } c \leqslant y \leqslant 1 \end{aligned} \tag{5.1}$$

The cover (3.4) is used and the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is studied on the attractor of this system. The free energy functions of α_v and β_v and of Λ are obtained analytically. The entropy functions of α_v and β_v and of Λ are also obtained in analytic forms. We find the first-order phase transitions in the thermodynamics of α_y and of Λ. Zeros of the partition functions are studied in relation to the phase transitions. A scaling form of the mean value of α_y around a phase transition point is given.

5.1. Statistical Thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

The cover of natural size, $C(n, m)$ in (3.4), is isomorphic to Θ_K, $K = n + m$. In order to write the length of the interval $J(\sigma_1 \sigma_2 \cdots \sigma_k)$ and the probability measure $P_Y(\sigma_1 \sigma_2 \cdots \sigma_k)$, we use the mapping V conjugate to Y:

$$V(u) = \begin{cases} 2u & \text{for } \ 0 \leqslant u < 1/2 \\ 2u - 1 & \text{for } \ 1/2 \leqslant u \leqslant 1 \end{cases} \tag{5.2}$$

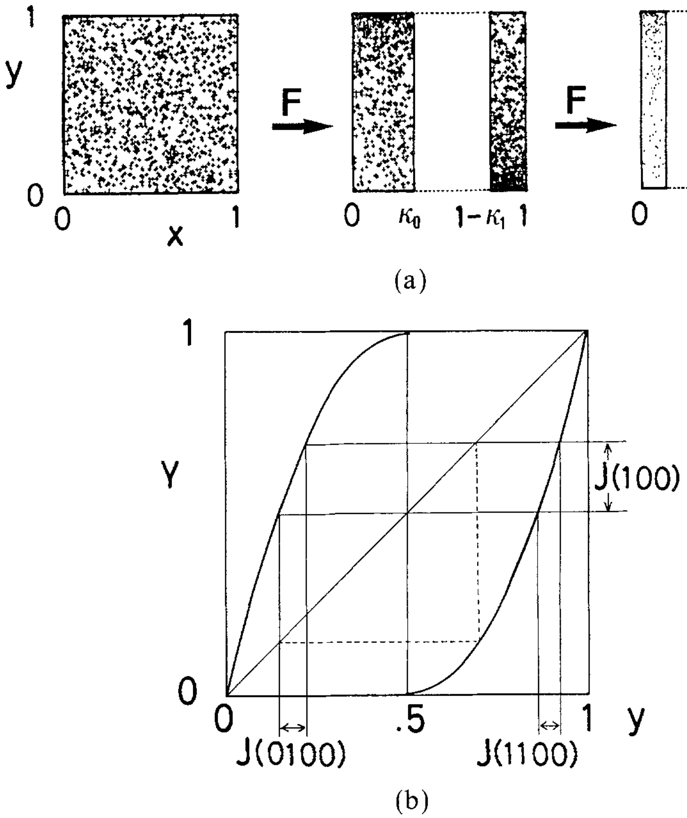


Fig. 6. (a) Schematic illustrating the mapping (5.1). (b) The one-dimensional map Y in (5.1).

The conjugacy from V to Y is

$$y = \sin^2(\pi u/2) \tag{5.3}$$

Using the binary expansion of $u \in [0, 1]$, we write

$$J(\sigma_1 \sigma_2 \cdots \sigma_k) = \left(\sin^2\left[\frac{\pi}{2}\sum_{j=1}^{k} \sigma_j 2^{-j}\right], \sin^2\left[\frac{\pi}{2}\left(2^{-k} + \sum_{j=1}^{k} \sigma_j 2^{-j}\right)\right]\right)$$
$$\text{except for all } \sigma_j = 1 \tag{5.4}$$

Since the natural measure of V is the length of an interval, we have

$$P_Y(\sigma_1 \sigma_2 \cdots \sigma_k) = 2^{-k} \tag{5.5}$$

The dynamical partition of equal mass leads to

$$\alpha_x(\boldsymbol{\sigma}_x)\,\beta_x(\boldsymbol{\sigma}_x)=\alpha_y(\boldsymbol{\sigma}_y)\,\beta_y(\boldsymbol{\sigma}_y)=\ln 2 \tag{5.6}$$

The partition function is written as

$$\begin{aligned}\Xi_{nm}(q,\boldsymbol{\tau})=&\,[(1/\kappa_0)^{\tau_x}+(1/\kappa_1)^{\tau_x}]^n\,e^{-(n+m)q\ln 2}\\&\times\left\{[\sin(2^{-m-1}\pi)]\sum_{\{\boldsymbol{\sigma}_y\}}\sin\left[\left(2^{-m-1}+\sum_{k=2}^{m}\sigma_{k+n}2^{-k}\right)\pi\right]\right\}^{-\tau_y}\end{aligned} \tag{5.7}$$

The free energy function of α_x and β_x turns out to be

$$G_x(q,\tau_x)=q\ln 2-\ln[(1/\kappa_0)^{\tau_x}+(1/\kappa_1)^{\tau_x}] \tag{5.8}$$

Changing the summation in (5.7) for the integration, we get

$$G_y(q,\tau_y)=(q-1-\tau_y)\ln 2-\lim_{m\to\infty}\frac{1}{m}\ln\left|\frac{1-\exp[-m(1-\tau_y)\ln 2]}{1-\tau_y}\right| \tag{5.9}$$

The limit in (5.9) yields the nonanalyticity of $G_y(q,\tau_y)$ at $\tau_y=1$. Equation (5.7) can be written as

$$\begin{aligned}\Xi_{nm}(q,\boldsymbol{\tau})=&\left[\left(\frac{1}{\kappa_0}\right)^{\tau_x}+\left(\frac{1}{\kappa_1}\right)^{\tau_x}\right]^n\exp[-(n+m)\,q\ln 2]\\&\times B\frac{1}{1-\tau_y}\{1-\exp[-m(1-\tau_y)\ln 2]\}\exp[m(1+\tau_y)\ln 2]\end{aligned} \tag{5.10}$$

for sufficiently large m, where $B\sim O(1)$. The form of $f_x(\alpha_x)$ is the same as that obtained for the generalized baker's transformation.[3] One can obtain $f_y(\alpha_y)$ from the inverse Laplace transform of (5.10). Actually, the entropy function of β becomes

$$\begin{aligned}s_{\beta;\,y}(\beta)&=\lim_{m\to\infty}\frac{1}{m}\ln\left[\frac{m}{2\pi i}\int_{-i\infty}^{+i\infty}dz\,\frac{1-e^{-m(1-z)\ln 2}}{1-z}\,e^{-mz(\beta-\ln 2)}\right]\\&=-\beta+\ln 2\qquad\text{for}\quad \ln 2<\beta<\ln 4\end{aligned} \tag{5.11}$$

Using (5.6) and $\beta_y f_y(\alpha_y)=s_{\beta;\,y}(\beta_y)+\alpha_y$, we get[12]

$$f_y(\alpha_y)=2\alpha_y-1\qquad\text{for}\quad 1/2\leqslant\alpha_y\leqslant 1 \tag{5.12}$$

The definition of $\tau^*(q)$ leads to

$$\exp(-\tau_x^* \ln \kappa_0) + \exp(-\tau_x^* \ln \kappa_1) = 2^q \tag{5.13a}$$

$$\tau_y^*(q) = \begin{cases} q-1 & \text{for} \quad q \leqslant 2 \\ q/2 & \text{for} \quad q > 2 \end{cases} \tag{5.13b}$$

We see the first-order phase transition in the thermodynamics of α_y from (5.9) or (5.13b), while the thermodynamics of α_x has no phase transition. In fact, the mean and variance of α_y are written for sufficiently large m as (see Fig. 7)

$$\begin{aligned} \langle \alpha_y \rangle (q, \tau_y; m) &= (\ln 2) \langle \beta_y^{-1} \rangle (q, \tau_y; m) \\ &\simeq \frac{x}{e^x - 1} e^{-x} \int_{-\infty}^{x} dz \, \frac{e^z - 1}{z} e^z \qquad \text{with} \quad x \equiv m(\tau_y - 1) \ln 2 \end{aligned} \tag{5.14a}$$

$$= \begin{cases} \dfrac{1}{2} + \dfrac{1}{4x} + \cdots & \text{for} \quad x > 1 \\ \ln 2 + \left(1 - \dfrac{3}{2} \ln 2\right) x + \cdots & \text{for} \quad |x| < 1 \\ 1 - \dfrac{1}{|x|} + \cdots & \text{for} \quad x < -1 \end{cases} \tag{5.14b}$$

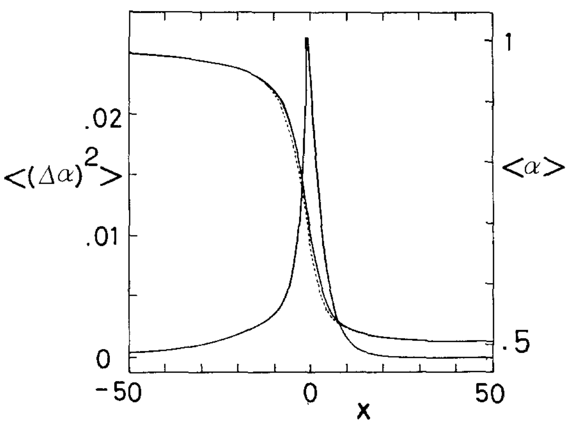


Fig. 7. The order parameter α_y shows a first-order phase transition at $\tau_y = 1$. $\langle \alpha_y \rangle$ and $\langle (\Delta\alpha_y)^2 \rangle$ have scaling forms with $x = m(\tau_y - 1) \ln 2$. The dotted line denotes $(\ln 2)/\langle \beta_y \rangle$, where $\langle \beta_y \rangle$ has the same scaling form with x as $\langle \Lambda_1 \rangle$ of (B.5). The scaling form of $\langle \alpha_y \rangle$ is different from $(\ln 2)/\langle \beta_y \rangle$.

$$
\begin{aligned}
&\langle (\Delta\alpha_y)^2\rangle(q, \tau_y; m) \\
&\quad \equiv \frac{(m \ln 2)^2}{\Xi_{nm}(q, \tau_x, \tau_y)} \int_{-\infty}^{\tau_y} dt \int_{-\infty}^{t} \Xi_{nm}(q, \tau_x, z)\, dz - \langle \alpha_y \rangle^2 \\
&\quad \simeq \left(\langle \alpha_y \rangle - \frac{1}{2} \frac{2-e^x}{1-e^x} \right) x - \langle \alpha_y \rangle^2 \\
&\quad = \begin{cases} O(x^{-2}) & \text{for} \quad |x| \gg 1 \\ \frac{1}{2} - (\ln 2)^2 + \left[3(\ln 2)^2 - \ln 2 - \frac{3}{4} \right] x & \\ \quad - \left[\frac{53}{12} (\ln 2)^2 - 3 \ln 2 - \frac{1}{24} \right] x^2 + \cdots & \text{for} \quad |x| < 1 \end{cases}
\end{aligned} \tag{5.15}
$$

As m tends to infinity, $\langle \alpha_y \rangle$ becomes discontinuous at $\tau_y = 1$. $m\langle (\Delta\alpha_y)^2 \rangle$, the mean squared fluctuation of α_y, grows to the size m of system for $\tau_y \to 1$, while it decays proportional to m^{-1} out of the transition region. Relations (5.14) and (5.15) show that a finite-size scaling law for the first-order phase transition holds. It is easy to calculate the scaling forms for the mean and variance of β_y. Note that the scaling forms of $\langle \alpha_y \rangle$ and of $(\ln 2)/\langle \beta_y \rangle$ are different.

Let us consider the interaction of α_y and β_y:

$$
\begin{aligned}
&qU_y(\boldsymbol{\sigma}_y; q, \tau_y) \\
&\quad \equiv mq\alpha_y(\boldsymbol{\sigma}_y)\, \beta_y(\boldsymbol{\sigma}_y) - m\tau_y \beta_y(\boldsymbol{\sigma}_y) \\
&\quad = mq \ln 2 + \tau_y \ln |\sin(2^{-m-1}\pi)| \\
&\qquad + \tau_y(1 - \sigma_{1+n}) \ln \left| \sin \left[\left(2^{-m-1} + \sum_{k=2}^{m} \sigma_{k+n} 2^{-k} \right) \pi \right] \right| \\
&\qquad + \tau_y \sigma_{1+n} \ln \left| \sin \left\{ \left[2^{-m-1} + \sum_{k=2}^{m} (1 - \sigma_{k+n})\, 2^{-k} \right] \pi \right\} \right|
\end{aligned} \tag{5.16}
$$

The first two terms of (5.16) are independent of which symbol, 0 or 1, occupies each lattice point. The last two terms, denoted by ΔU, depend on the configuration of symbols occupying all lattice points. Almost every configuration of symbols has $\Delta U \sim O(1) \times \tau_y$. The configuration that identical symbols successively occupy from the $(1+n)$th lattice point to the $(k+n)$th, called a cluster of size k, has $\Delta U \sim k\tau_y \ln 2$; these terms correspond to the square root divergence of the probability density on the attractor of Y at $y=0$ and $y=1$. However, a cluster of size k has $\Delta U \sim O(1) \times \tau_y$, when the symbol at the $(1+n)$th lattice point is different from the symbol of the cluster. Thus, the system is nonuniform. For $\tau_y > 1$, clusters growing from the $(1+n)$th lattice point up to the size m of the system overcome all other configurations and dominantly contribute to the free energy function. For $\tau_y < 1$, the contribution to the free energy comes equally from all configurations, so that the number of configurations becomes important. Similar facts were pointed out by Katzen and Procaccia in their study of the attractor of the fully developed logistic map.[17]

5.2. Local Lyapunov Exponents

Let us consider the local Lyapunov exponents given by (2.10). We write Λ_2 determined during a time interval k $(k \leqslant m)$ as

$$\Lambda_2(\boldsymbol{\sigma}_y; k) = \frac{1}{k} \sum_{j=1}^{k} \ln \kappa(\sigma_{j+n}) \tag{5.17}$$

when an initial value of the reference orbit belongs to $\Omega_{nm}(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$. On the other hand, two reference orbits which have initial values of different y_0 give different Λ_1 even if the initial values belong to $\Omega_{nm}(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$. We consider

$$\tilde{\Lambda}_1(\boldsymbol{\sigma}_y; k) \equiv \frac{1}{k} \ln \left[\frac{l_y(\sigma_{k+1+n}\sigma_{k+2+n} \cdots \sigma_{m+n})}{l_y(\sigma_{1+n}\sigma_{2+n} \cdots \sigma_{m+n})} \right] \tag{5.18}$$

As $m \to \infty$, $\tilde{\Lambda}_1(\boldsymbol{\sigma}_y; k)$ tends to $\Lambda_1(\mathbf{x}_0; k)$ for $\mathbf{x}_0 \in \Omega_{nm}(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)$. We study the statistical thermodynamics of $\tilde{\Lambda}_1$ and Λ_2 where the Gibbs ensemble and the partition function are given by

$$\begin{aligned} \tilde{p}&\left(\boldsymbol{\sigma}_y; z_1, z_2, \frac{m}{k} = r\right) \\ &\equiv \frac{1}{\Xi_k(z_1, z_2; r)} P_Y(\boldsymbol{\sigma}_y) \\ &\qquad \times \exp\{-k[z_1 \tilde{\Lambda}_1(\boldsymbol{\sigma}_y; k) + z_2 \Lambda_2(\boldsymbol{\sigma}_y; k)]\} \end{aligned} \tag{5.19}$$

$$\begin{aligned} \Xi_k&(z_1, z_2; r) \\ &\equiv \sum_{\{\sigma_y\}} P_Y(\boldsymbol{\sigma}_y) \exp\{-k[z_1 \tilde{\Lambda}_1(\boldsymbol{\sigma}_y; k) + z_2 \Lambda_2(\boldsymbol{\sigma}_y; k)]\} \end{aligned} \tag{5.20}$$

The free energy function of $\tilde{\Lambda}_1$ and Λ_2 is defined by

$$\tilde{\Phi}(z_1, z_2; r) \equiv -\lim_{k \to \infty} \frac{1}{k} \ln \Xi_k(z_1, z_2; r) \qquad \text{with } \frac{m}{k} = r \text{ fixed} \tag{5.21}$$

For k and $r \gg 1$, (5.20) can be written as

$$\Xi_k(z_1, z_2; r) \simeq 2^{-m_2-kz_1} \sum_{i=2}^{k} \sum_{\{\sigma_{i+1} \cdots \sigma_m\}} \prod_{n=i+1}^{k} \left[\frac{1}{\kappa(\sigma_n)}\right]^{z_2} \times \left\{ \left[\left(\frac{1}{\kappa_0}\right)^{i-1} \frac{1}{\kappa_1}\right]^{z_2} \left| \frac{\sin[2^{-i}\pi + \zeta(\{\sigma\}; i, m)]}{\sin[2^k \zeta(\{\sigma\}; k, m)]} \right|^{z_1} + \left[\left(\frac{1}{\kappa_1}\right)^{i-1} \frac{1}{\kappa_0}\right]^{z_2} \left| \frac{\sin[2^{-i}\pi + \zeta(\{\bar{\sigma}\}; i, m)]}{\sin[2^k \zeta(\{\bar{\sigma}\}; k, m)]} \right|^{z_1} \right\} \tag{5.22}$$

where $\bar{\sigma} \equiv 1 - \sigma$ and

$$\zeta(\{\sigma\}; i, m) \equiv \left(2^{-m-1} + \sum_{j=i+1}^{m} \sigma_j 2^{-j}\right)\pi$$

The Jordan inequality leads to the following inequalities:

$$2^{-(i-1)} < \sin[2^{-i}\pi + \zeta(\{\sigma\}; i, m)] < \pi 2^{-(i-1)}$$

Using these inequalities in (5.22), we get

$$\Gamma \lesseqgtr \Xi_k(z_1, z_2; r) \lesseqgtr \pi^{z_1}\Gamma \qquad \text{for} \quad z_1 \gtreqless 0, \text{ respectively} \tag{5.23}$$

where we put

$$\begin{aligned} \Gamma &\equiv 2^{-m_2-kz_1} \sum_{i=2}^{k} \left[\left(\frac{1}{\kappa_0}\right)^{z_2} + \left(\frac{1}{\kappa_1}\right)^{z_2}\right]^{k-i} \\ &\quad \times \left[\left(\frac{1}{\kappa_0}\right)^{iz_2} \left(\frac{\kappa_0}{\kappa_1}\right)^{z_2} + \left(\frac{1}{\kappa_1}\right)^{iz_2} \left(\frac{\kappa_1}{\kappa_0}\right)^{z_2}\right] \\ &\quad \times 2^{-(i-1)z_1} \left(2 \sum_{j=2}^{m-k} 2^{m-k-j} 2^{(j-1)z_1}\right) \\ &\simeq \xi\eta \left(\frac{\eta}{2}\right)^k \left(\frac{1}{\kappa_0}\right)^{kz_2} \left[\frac{\eta^{k-1} - (1+\xi)^{k-1}}{\eta - (1+\xi)} + \frac{(\xi\eta)^{k-1} - (1+\xi)^{k-1}}{\xi\eta - (1+\xi)}\right] \\ &\quad \times \frac{1 - (2\eta)^{-k(r-1)}}{2\eta - 1} \end{aligned} \tag{5.24}$$

with $\xi = (\kappa_0/\kappa_1)^{z_2}$ and $\eta = 2^{-z_1}$. Without loss of generality, we assume $\kappa_0 > \kappa_1$. Then, the following result is obtained:

$$\begin{aligned}\tilde{\Phi}(z_1, z_2; r) &\\ &= -(r-1)(z_1-1)\ln 2 + (z_1+1)\ln 2 \\ &\quad - \ln[(1/\kappa_0)^{z_2} + (1/\kappa_1)^{z_2}] && \text{in region A} \quad (5.25a)\\ &= (z_1+1)\ln 2 - \ln[(1/\kappa_0)^{z_2} + (1/\kappa_1)^{z_2}] && \text{in region B} \quad (5.25b)\\ &= (2z_1+1)\ln 2 + z_2 \ln \kappa_0 && \text{in region C} \quad (5.25c)\\ &= (2z_1+1)\ln 2 + z_2 \ln \kappa_1 && \text{in region D} \quad (5.25d)\end{aligned}$$

where regions A–D are given in Fig. 8. In the limit $r \to \infty$, $\tilde{\Phi}(z_1, z_2; r)$ yields the free energy function of Λ:

$$\Phi(z_1, z_2) = \lim_{r \to \infty} \tilde{\Phi}(z_1, z_2; r) \tag{5.26}$$

The results (5.25a)–(5.25d) show the existence of phase transitions in the thermodynamics of Λ_1 and Λ_2. The phase diagram of Λ_1 and Λ_2 is described in Fig. 8. All of the transitions are of first order. In region A, the main contribution to the free energy comes from the reference orbits the last point of which falls into the critical line $y = c$ where the derivative of Y vanishes. The average value of Λ_1, denoted by $\langle \Lambda_1 \rangle$, tends to negative

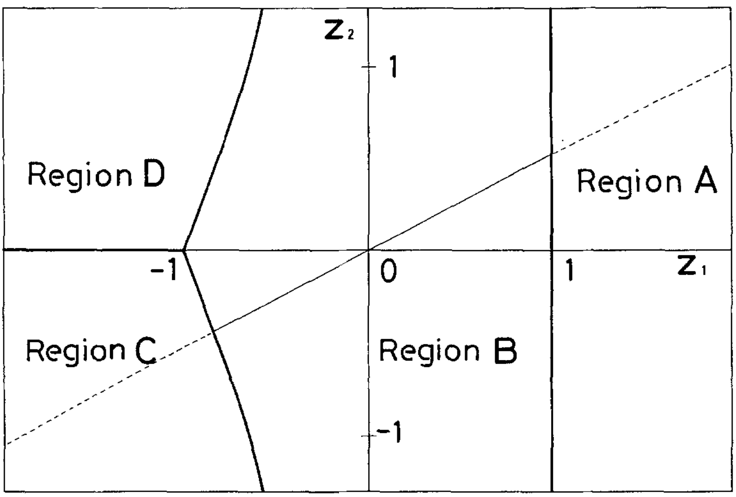


Fig. 8. Phase diagram of local Lyapunov exponents on the attractor of (5.1). The heavy lines denote the phase boundaries. Regions A–D are shown. On the light line in region B, $\Phi(\tau_y^*, \tau_x^*) = 0$ holds. The parameter values are $\kappa_0 = 0.35$ and $\kappa_1 = 0.2$.

infinity. When z_1 decreases and goes across the phase boundary of $z_1 = 1$, $\langle \Lambda_1 \rangle$ jumps from $-\infty$ to ln 2 at the phase boundary.

In region B, the typical chaotic orbits contribute to the free energy dominantly and the local Lyapunov exponents behave as on a hyperbolic chaotic attractor. In fact, we see that the relations (1.1) and (1.2) hold in region B: From (5.8), we have $\Psi(q) = (q-1)\ln 2$. Since (5.25b) gives $\Phi(q-1, 0) = (q-1)\ln 2$, (1.2) holds in region B. It turns out from (5.13) that the curve given by $z_1 = \tau_y^*(q)$ and $z_2 = \tau_x^*(q)$ is written as

$$z_1 = \frac{1}{\ln 2} \ln \left[\left(\frac{1}{\kappa_0} \right)^{z_2} + \left(\frac{1}{\kappa_1} \right)^{z_2} \right] - 1 \qquad \text{for} \quad z_1 < 1 \tag{5.27}$$

On this curve, it turns out from (5.25b) that

$$\Phi(\tau_y^*, \tau_x^*) = 0 \qquad \text{in region B} \tag{5.28}$$

Through region B, $\langle \Lambda_1 \rangle(z_1, z_2) = \ln 2$, while $\langle \Lambda_2 \rangle$ changes smoothly. When z_1 decreases and goes across the phase boundary

$$z_1 = \begin{cases} -\dfrac{1}{\ln 2} \ln \left[1 + \left(\dfrac{\kappa_0}{\kappa_1} \right)^{z_2} \right] & \text{for} \quad z_2 \leqslant 0 \\ -\dfrac{1}{\ln 2} \ln \left[1 + \left(\dfrac{\kappa_1}{\kappa_0} \right)^{z_2} \right] & \text{for} \quad z_2 > 0 \end{cases} \tag{5.29}$$

$\langle \Lambda_1 \rangle$ jumps from ln 2 to ln 4 at the phase boundary; $\langle \Lambda_2 \rangle$ also has a jump there. These discontinuities lead to the breakdown of (1.1) and (1.2) in regions C and D.

In regions C and D, $\langle \Lambda_1 \rangle(z_1, z_2) = \ln 4$. The main contribution to the free energy comes from the reference orbits which stay near the fixed point $\mathbf{x}^*$ of $\mathbf{F}$ during the time interval k: $\mathbf{x}^* = (0, 0)$ in region C and (1, 1) in region D. We have $\langle \Lambda_2 \rangle(z_1, z_2) = \ln \kappa_0$ in region C and $\ln \kappa_1$ in region D. When z_2 goes across the phase boundary of $z_2 = 0$ from region C to region D, $\langle \Lambda_2 \rangle$ jumps from $\ln \kappa_0$ to $\ln \kappa_1$ at $z_2 = 0$. When calculating the fluctuations of the local Lyapunov exponents by using the partition function (5.24), we observe the enhancement of the fluctuations near the phase boundaries. In Appendix B, a scaling form of $\langle \Lambda_1 \rangle$ near the phase boundary (5.29) is studied.

The entropy function $s_\Lambda(\tilde{\Lambda}_1, \Lambda_2)$ may be calculated from the Legendre–Fenchel transformation of $\Phi(z_1, z_2)$. The calculation of $s_\Lambda(\tilde{\Lambda}_1, \Lambda_2)$ is given in Appendix A, where $s_\Lambda(\tilde{\Lambda}_1, \Lambda_2)$ is derived directly from the partition function (5.24). On the assumption of $\kappa_0 > \kappa_1$, the following result is obtained:

$$
\begin{aligned}
s_\Lambda(\tilde{\Lambda}_1, \Lambda_2; r) & \\
&= -\infty \quad \text{for} \quad \ln \kappa_1 > \Lambda_2, \ln \kappa_0 < \Lambda_2, \\
&\qquad\qquad \tilde{\Lambda}_1 < (2-r)\ln 2, \text{ or } \tilde{\Lambda}_1 > \ln 4 \qquad (5.30a) \\
&= \tilde{\Lambda}_1 - \ln 4 - y_1 \ln y_1 - y_2 \ln y_2 \\
&\quad \text{for} \quad \ln \kappa_1 < \Lambda_2 < \ln \kappa_0 \text{ and } (2-r)\ln 2 < \tilde{\Lambda}_1 < \ln 2 \qquad (5.30b) \\
&= \varphi(y_1, y_2) \quad \text{for} \quad \tfrac{1}{2}\ln(\kappa_0\kappa_1) \leqslant \Lambda_2 < \ln \kappa_0, \\
&\qquad\qquad \ln 2 < \tilde{\Lambda}_1 < \ln 4, \text{ and } y_1 > y_* \qquad (5.30c) \\
&= \varphi(y_2, y_1) \quad \text{for} \quad \tfrac{1}{2}\ln(\kappa_0\kappa_1) > \Lambda_2 > \ln \kappa_1, \\
&\qquad\qquad \ln 2 < \tilde{\Lambda}_1 < \ln 4, \text{ and } y_2 > y_* \qquad (5.30d)
\end{aligned}
$$

where y_1, y_2, y_*, and $\varphi(y_1, y_2)$ are given by

$$y_1 = 1 - y_2 = \frac{1}{\ln(\kappa_0/\kappa_1)}(\Lambda_2 - \ln \kappa_1) \qquad (5.31)$$

$$y_* = \frac{\tilde{\Lambda}_1}{\ln 2} - 1 \qquad (5.32)$$

$$\varphi(y_1, y_2) = -\ln 2 - y_2 \ln y_2 - (y_1 - y_*) \ln |y_1 - y_*| + (1 - y_*) \ln |1 - y_*| \qquad (5.33)$$

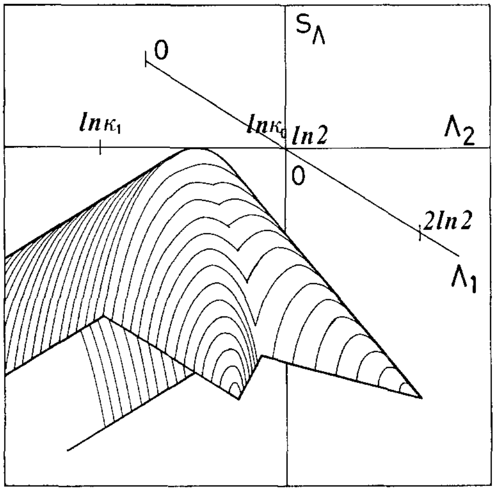


Fig. 9. The entropy function $s_\Lambda(\Lambda_1, \Lambda_2)$ of local Lyapunov exponents on the attractor of (5.1). The maximum value of $s_\Lambda(\Lambda_1, \Lambda_2)$ is zero at $\Lambda_1 = \ln 2$ and $\Lambda_2 = \frac{1}{2}\ln(\kappa_0\kappa_1)$. The parameter values are the same as in Fig. 8.

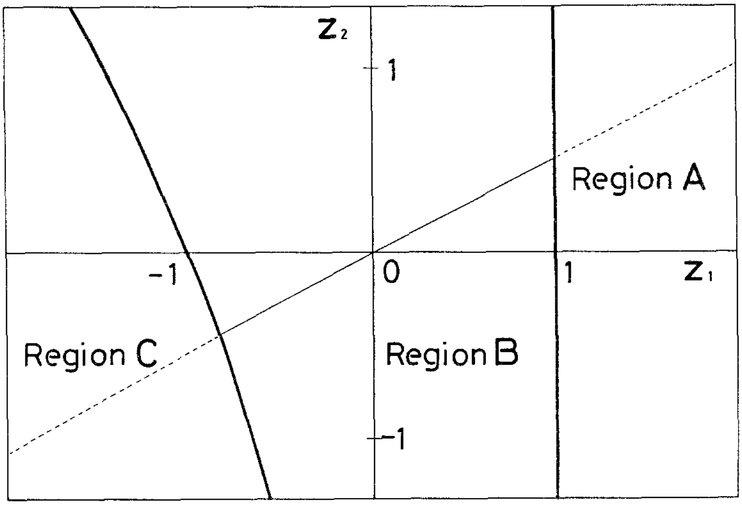


Fig. 10. Phase diagram of local Lyapunov exponents on the attractor of $\tilde{\mathbf{F}}$. There exists no region corresponding to region D of Fig. 8. The parameter values are the same as in Fig. 8.

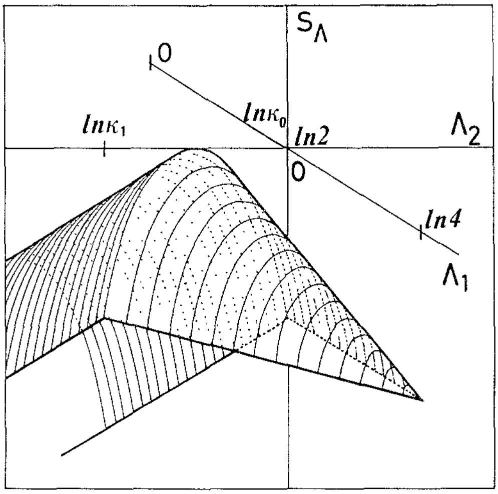


Fig. 11. The entropy function of local Lyapunov exponents on the attractor of $\tilde{\mathbf{F}}$. The parameter values are the same as in Fig. 8.

$s_\Lambda(\Lambda_1, \Lambda_2)$ is obtained from $s_\Lambda(\tilde{\Lambda}_1, \Lambda_2; r)$ in the limit $r \to \infty$ and is shown in Fig. 9. Note that $s_\Lambda(\tilde{\Lambda}_1, \Lambda_2; r)$ of (5.30) is different from the Legendre–Fenchel transform of $\Phi(z_1, z_2)$, which is the convex hull of (5.30).

Let us consider the map $\tilde{\mathbf{F}}$ which consists of the same $X(x, y)$ as (5.1) and $Y(y) = 4y(1-y)$. The statistical thermodynamics of $\mathbf{\Lambda}$ is different between the maps $\mathbf{F}$ and $\tilde{\mathbf{F}}$, while the statistical thermodynamics of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is identical between them.[30] $\mathbf{F}$ has the fixed points (0, 0) and (1, 1). On the other hand, $\tilde{\mathbf{F}}$ has the fixed points (0, 0) and (1, 3/4). A reference orbit staying near (0, 0) or (1, 1) of $\mathbf{F}$ during the time interval k makes an important contribution to the free energy of $\mathbf{\Lambda}$. However, one staying near (1, 3/4) of $\tilde{\mathbf{F}}$ has a contribution of the same order as almost every orbit, so that this contribution is less important. An orbit of $\tilde{\mathbf{F}}$ passing near (1, 1) falls into the neighborhood of the y axis and stays near (0, 0) for a long time. Hence, there is no region D in the phase diagram of $\Phi(z_1, z_2)$ on the attractor of $\tilde{\mathbf{F}}$. Figures 10 and 11 display the phase diagram of $\Phi(z_1, z_2)$ and the entropy function $s_\Lambda(\Lambda_1, \Lambda_2)$ on the attractor of $\tilde{\mathbf{F}}$.

5.3. Zeros of a Partition Function

We have obtained the analytic forms of the partition functions, so that the distribution of zeros of the partition functions can be studied easily. In order to study the zeros of (5.20), we use (5.24). We have

$$\Xi_k(\eta, \xi; r) \sim \Xi_k^0(\eta, \xi; r) \left[\frac{\eta^{k-1} - (1+\xi)^{k-1}}{\eta - (1+\xi)} + \frac{(\xi\eta)^{k-1} - (1+\xi)^{k-1}}{\xi\eta - (1+\xi)} \right] \frac{1 - (2\eta)^{-k(r-1)}}{2\eta - 1}$$

$\Xi_k^0(\eta, \xi; r)$ has zeros at the origins of the complex η and ξ planes. The zeros of $\Xi_k(\eta, \xi; r)$ do not lie on the positive real axis in the complex η plane or in the complex ξ plane.[42] In the complex η plane, $m-k-1$ zeros are distributed uniformly on the circle $|\eta| = 1/2$ and $k-2$ zeros on the circle $|\eta| = 1+\xi$ if $\xi < 1$ and $|\eta| = 1+\xi^{-1}$ if $\xi \geqslant 1$, for sufficiently large k. As k tends to infinity with $m/k = r$ fixed, the zeros lying near the points $\eta = 1/2$ and $\eta = 1+\xi$ or $1+\xi^{-1}$ fall into these points, respectively. Then, the analyticity of $\Phi(z_1, z_2)$ breaks down at $z_1 = 1$ and the phase boundary (5.29).

In the ξ plane, the distribution of the zeros is rather complicated. We examine roots of the algebraic equation of ξ:

$$(\eta - 2)(1+\xi)^k + \eta^{k-1}(\xi + 1 - \eta)\, \xi^{k-1} + \eta^{k-1}(\xi + 1 - \eta\xi) = 0 \qquad (5.34)$$

$\xi = \eta - 1$ and $\xi = 1/(\eta - 1)$ are roots of (5.34), but not zeros of $\Xi_k(\eta, \xi; r)$. For $z_1 < -1$ (i.e., $\eta > 2$), the other roots of (5.34) lie on the circle $|\xi| = 1$ in the thermodynamic limit $k \to \infty$: ξ_j $(j = 1, 2, ..., k-2)$ denotes a root of (5.34),

$$\xi_j \simeq \exp(i\theta_j) \quad \text{where} \quad \theta_j = \frac{2\pi}{k-1}(j + \delta_j), \quad -\frac{1}{2} < \delta_j \leqslant \frac{1}{2} \tag{5.35}$$

For $z_1 > -1$, the $k-2$ roots of (5.34) lie on the arcs of three circles in the limit $k \to \infty$: The $k-2$ roots are

$$\begin{aligned} \xi_j &\simeq \exp(i\theta_j) && \text{if} \quad 0 \leqslant \cos(\theta_j/2) < |\eta - 1|/2 \\ 1 + \xi_j &\simeq \eta \exp(i\theta_j) && \text{if} \quad \cos(\theta_j) > |\eta - 1|/2 \\ 1 + \xi_j^{-1} &\simeq \eta \exp(i\theta_j) && \text{if} \quad \cos(\theta_j) > |\eta - 1|/2 \end{aligned} \tag{5.36}$$

where $\theta_j = 2\pi(j + \delta_j)/(k-1)$ with $-\frac{1}{2} < \delta_j \leqslant \frac{1}{2}$. The distributions of the zeros of $\Xi_k(\eta, \xi; r)$ in the limit $k \to \infty$ are shown in Fig. 12. In the limit $k \to \infty$, the zeros lie on the points in the positive real ξ axis, i.e., $\xi = 1$ for $z_1 < -1$ and $\xi = \eta - 1$ and $\xi = 1/(\eta - 1)$ for $-1 < z_1 < 0$. This leads to the phase transition of Λ_1 and Λ_2 at each transition point. On the other hand, all roots of (5.36) lie on the left half-plane of Re $\xi < 0$ for $z_1 > 0$. Thus, no phase transition occurs for $z_1 > 0$ when z_2 changes.

6. DISCUSSION AND SUMMARY

The example in Section 5 shows that (1.1) and (1.2), which hold on a hyperbolic chaotic attractor, hold only within a restricted region of q for a nonhyperbolic chaotic attractor. We reexamine the relation (1.1) and investigate why (1.1) holds in region B and does not hold in the other regions.[14] The definition of τ_v^*, i.e., (3.23), yields

$$\begin{aligned} \Xi_{nm}(q, \tau_x^*, \tau_y^*) = \sum_{\{\sigma\}} & [P(\boldsymbol{\sigma}_x; \boldsymbol{\sigma}_y)]^q \exp\{-\tau_x^* \ln[l_x(\boldsymbol{\sigma}_x)] \\ & - \tau_y^* \ln[l_y(\boldsymbol{\sigma}_y)]\} \\ & \sim O(n^b, m^{b'}) \qquad \text{for sufficiently large } n \text{ and } m \end{aligned} \tag{6.1}$$

where b and b' are real and bounded. From (5.19) and (5.20), we have

$$\begin{aligned} l_x(\sigma_1 \cdots \sigma_{n+k}) &= l_x(\sigma_1 \cdots \sigma_n) \exp[k\Lambda_2(\boldsymbol{\sigma}_y; k)] \\ l_y(\sigma_{k+1+n} \cdots \sigma_{m+n}) &= l_y(\sigma_{1+n} \cdots \sigma_{m+n}) \exp[k\tilde{\Lambda}_1(\boldsymbol{\sigma}_y; k)] \end{aligned} \tag{6.2}$$

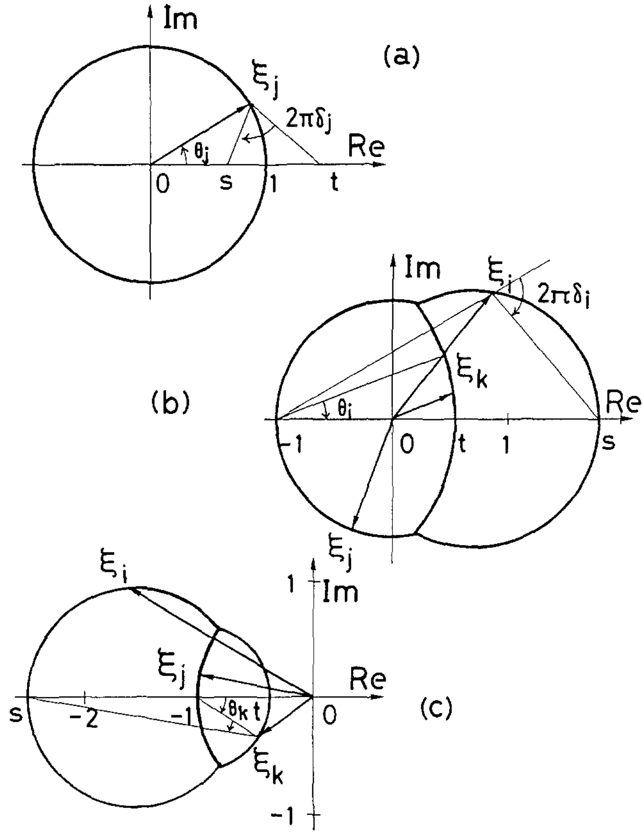


Fig. 12. The distribution of zeros of the partition function (5.24) on the complex ξ plane. The zeros are distributed densely on the arcs of the circles in the thermodynamic limit $k \to \infty$. The ξ_i, ξ_j, and ξ_k are zeros of (5.36), where $t = 1/s = \eta - 1$. The θ_i, θ_j, and θ_k are the arguments of ξ_i, ξ_j, and ξ_k, respectively. The δ_i, δ_j, and δ_k are the deviations of θ_i, θ_j, and θ_k from the uniform distributions. Shown are the cases of (a) $z_1 < -1$, (b) $-1 < z_1 < 0$, and (c) $0 < z_1$.

Combining (6.1) and (6.2), we write

$$\begin{aligned}\Xi_{n+k,m-k}(q, \tau_x^*, \tau_y^*) = \sum\, &[P_Y(\boldsymbol{\sigma})]^q \exp\{-\tau_x^* \ln[l_x(\boldsymbol{\sigma}_x)] \\ &- \tau_y^* \ln[l_y(\boldsymbol{\sigma}_y)]\} \\ &\times \exp\{-k[\tau_y^* \tilde{\Lambda}_1(\boldsymbol{\sigma}_y; k) + \tau_x^* \Lambda_2(\boldsymbol{\sigma}_y; k)]\} \\ \sim\, &O(n^b, m^{b'}, k^{b''}) \end{aligned} \tag{6.3}$$

Assuming the Markov partition (3.17), we have

$$\sum_{\{\boldsymbol{\sigma}_y\}} \left([P_Y(\boldsymbol{\sigma}_y)]^{q-1} \exp\{-\tau_y^* \ln[l_y(\boldsymbol{\sigma}_y)]\} \times P_Y(\boldsymbol{\sigma}_y) \exp\{-k[\tau_y^* \tilde{\Lambda}_1(\boldsymbol{\sigma}_y; k) + \tau_x^* \Lambda_2(\boldsymbol{\sigma}_y; k)]\}\right) \sim O(m^b, k^{b'}) \tag{6.4}$$

If it holds that

$$[P_Y(\boldsymbol{\sigma}_y)]^{q-1} \sim \exp\{\tau_y^* \ln[l_y(\boldsymbol{\sigma}_y)]\} \quad \text{for a.e. } \boldsymbol{\sigma}_y \quad \text{with respect to } \tilde{\rho}(\boldsymbol{\sigma}_y; \tau_y^*, \tau_x^*, r) \text{ of (5.19)} \tag{6.5}$$

then (6.4) yields

$$\tilde{\Phi}(\tau_y^*, \tau_x^*; r) = 0 \tag{6.6}$$

In fact, (6.5) holds in region B of (5.25), called the hyperbolic phase, as well as in the examples of Section 4. In the other regions of (5.25), the probability of states of particular configurations becomes finite and that of all states that

$$[P_Y(\boldsymbol{\sigma}_y)]^{q-1} = 2^{-m(q-1)} \sim \exp\{\tau_y^* \ln[l_y(\boldsymbol{\sigma}_y)]\}$$

vanishes as k tends to infinity.

The availability of the relation (1.1) for a nonhyperbolic chaotic attractor has been pointed out by Hata *et al.*[27] They assert that for a nonhyperbolic chaotic attractor of a two-dimensional map with constant Jacobian, if transition points of phase transitions in the thermodynamics of Λ_1 are known, then phase transitions occur in the thermodynamics of $\boldsymbol{\alpha}$ and these transition points are determined by using (1.1). In the system (5.1), the nonanalytic point $z_1 = 1$ of $\Phi(z_1, z_2)$ corresponds to the transition point $q = 2$ of α_y in relation (5.28), but there exists no transition point of α_y corresponding to the point on the phase boundary (5.29). This discrepancy may be traceable to the fact that the Jacobian of (5.1) is not constant.[12] However, one should note that (1.1) is generally derived under the assumption (6.5) and holds only in the hyperbolic phase.

In the examples of Section 4, we have seen that the following relation holds:

$$\Phi(z_1, z_2) = G_x(z_1 + 1, z_2) \tag{6.7}$$

On comparing (5.8) and (5.25b), it turns out that (6.7) holds in region B of the nonhyperbolic system (5.1) also. In order to study the relation (6.7), we can expand the same argument as in the above: If it holds that

$$[P_Y(\boldsymbol{\sigma}_y)]^{q-1} \exp\{-\tau_y \ln[l_y(\boldsymbol{\sigma}_y)]\} \sim \exp[-mG_y(q, \tau_y)] \quad \text{for a.e. } \boldsymbol{\sigma}_y \text{ with respect to } \tilde{\rho}(\boldsymbol{\sigma}_y; \tau_y, \tau_x, r) \text{ of (5.19)} \tag{6.8}$$

then we have

$$\tilde{\Phi}(\tau_y, \tau_x; r) = G_x(q, \tau_x) - G_y(q, \tau_y) \tag{6.9}$$

$\tilde{\Phi}(\tau_y, \tau_x; r)$ must be independent of q. Actually, inserting (5.8) and (5.9) into (6.9), we have

$$\tilde{\Phi}(\tau_y, \tau_x; r) = G_x(\tau_y + 1, \tau_x) \tag{6.10}$$

so that (5.25b) is obtained in the hyperbolic phase. Hence, one conjectures that not only (1.1) and (1.2), but also (6.7) holds in the hyperbolic phase of a chaotic attractor.

We have studied static and dynamic properties of a chaotic attractor of a two-dimensional map which belongs to a particular class of piecewise continuous invertible maps. Coverings of natural size to cover the attractor have been introduced, so that microscopic information of the attractor is written on each box composing the cover: The probability of a point in a box gives information on the natural measure, and the size distribution of the boxes describes the complexity of the geometric structure of the attractor. The statistical thermodynamics of the scaling indices and size indices of a box has been formulated in a natural way by using symbolic dynamics. The attractor turned out to have a generalized dimension which is the sum of generalized partial dimensions. Generalized entropies have also been studied and a relation between the generalized dimensions and entropies has been obtained. Illustrative examples, two of a hyperbolic chaotic attractor and one of a nonhyperbolic chaotic attractor, were studied. Analytic forms of the free energy functions of the scaling indices and size indices of a box have been obtained not only for the hyperbolic examples, but also for the nonhyperbolic example. For these examples, the statistical thermodynamics of local Lyapunov exponents has also been studied and a relation between the thermodynamics of scaling indices and of local Lyapunov exponents has been investigated. For the nonhyperbolic example, the free energy and entropy functions of local Lyapunov exponents have been obtained in analytic forms. These results display the existence of phase transitions in the thermodynamics of local Lyapunov exponents. A phase transition is seen in the thermodynamics of the scaling indices and size indices of a box also. Zeros of the partition function have been studied in relation to the violation of the analyticity of the free energy function. The zeros are distributed on arcs of circles in the thermodynamic limit.

APPENDIX A. DERIVATION OF $s_\Lambda(\tilde{\Lambda}_1, \Lambda_2; r)$

The entropy function of $\tilde{\Lambda}_1$ and Λ_2 is defined in terms of the probability density $W_\Lambda(\tilde{\Lambda}_1, \Lambda_2; k, r)$ as

$$s_\Lambda(\tilde{\Lambda}_1, \Lambda_2; r) \equiv \lim_{k \to \infty} \frac{1}{k} \ln W_\Lambda\left(\tilde{\Lambda}_1, \Lambda_2; k, \frac{m}{k} = r \text{ fixed}\right) \tag{A.1}$$

$W_\Lambda(\tilde{\Lambda}_1, \Lambda_2; k, r)$ describes the partition function (5.20) as follows:

$$\begin{aligned} \Xi_k(z_1, z_2; r) = \iint d\tilde{\Lambda}_1\, d\Lambda_2\, W_\Lambda(\tilde{\Lambda}_1, \Lambda_2; k, r) \\ \times \exp[-k(z_1 \tilde{\Lambda}_1 + z_2 \Lambda_2)] \end{aligned} \tag{A.2}$$

By using (5.24), one can write $\Xi_k(z_1, z_2; r)$ as

$$\begin{aligned} \Xi_k(z_1, z_2; r) = \sum_{i=2}^{m-k} \sum_{j=2}^{k} & \exp\{-[k(z_1+1) - i(z_1-1) + jz_1] \ln 2\} \\ & \times [\kappa_0^{-jz_2}(\kappa_0/\kappa_1)^{z_2} \\ & + \kappa_1^{-jz_2}(\kappa_1/\kappa_0)^{z_2}](\kappa_0^{-z_2} + \kappa_1^{-z_2})^{k-j} \end{aligned} \tag{A.3}$$

for $k \gg 1$. Let us divide (A.3) into two terms and consider the term

$$\begin{aligned} \sum_{i=2}^{m-k} \sum_{j=2}^{k} \sum_{n=1}^{k-j} \binom{k-j}{n} 2^{-(k+i)} \exp[-(k-i+j)\, z_1 \ln 2] \\ \times \exp[-(j-1+n)\, z_2 \ln \kappa_0 - (k-j-n+1)\, z_2 \ln \kappa_1] \end{aligned} \tag{A.4}$$

Comparing (A.2) and (A.4) leads to

$$\tilde{\Lambda}_1 = (1 - i/k + j/k) \ln 2 \tag{A.5a}$$

$$\Lambda_2 = \ln \kappa_1 + [(j-1)/k + n/k] \ln(\kappa_0/\kappa_1) \tag{A.5b}$$

For simplicity, assume $\kappa_0 > \kappa_1$. Then, we have

$$(2-r) \ln 2 \leqslant \tilde{\Lambda}_1 \leqslant \ln 4, \qquad \ln \kappa_1 \leqslant \Lambda_2 \leqslant \ln \kappa_0 \tag{A.6}$$

Using Stirling's approximation and replacing the summations with integrations, we can write (A.3) as

$$\frac{k^3}{b \ln 2} \int_0^1 dx \int_{bx + \ln \kappa_1}^{\ln \kappa_0} d\Lambda_2 \int_{(2-r+x)\ln 2}^{(1+x)\ln 2} d\tilde{\Lambda}_1$$
$$\times \exp[-kH_1(x; \tilde{\Lambda}_1, \Lambda_2) - k(z_1 \tilde{\Lambda}_1 + z_2 \Lambda_2)]$$

where $b \equiv \ln(\kappa_0/\kappa_1)$,

$$H_1(x; \tilde{\Lambda}_1, \Lambda_2) \equiv -\tilde{\Lambda}_1 + \ln 4 + x \ln 2$$
$$- (1-x)\ln(1-x) + (y_1 - x)\ln(y_1 - x) + y_2 \ln y_2 \qquad \text{(A.7)}$$

and y_1 and y_2 are given by (5.31). $H_1(x; \tilde{\Lambda}_1, \Lambda_2)$ is a monotone-increasing function of x on the interval $[0, y_1]$. Since $k \gg 1$, the maximum value approximation can be used in the integration of x. The contribution to the probability density from (A.3) becomes

$$W_1 \simeq \frac{k^2}{b \ln(2/y_1)} \exp[-kH_1(0; \tilde{\Lambda}_1, \Lambda_2)]$$
$$\text{for} \quad (2-r)\ln 2 < \tilde{\Lambda}_1 < \ln 2$$
$$\simeq \frac{k^2}{b \ln[(1-y_*)/(y_1 - y_*)]} \exp[-kH_1(y_*; \tilde{\Lambda}_1, \Lambda_2)]$$
$$\text{for} \quad \ln 2 \leqslant \tilde{\Lambda}_1 \leqslant \ln 4 \text{ and } y_1 > y_* \qquad \text{(A.8)}$$

where y_* is given by (5.32). The contribution of the other term to the probability density is calculated in the same way.

$$W_2 \simeq \frac{k^2}{b \ln(2/y_2)} \exp[-kH_2(0; \tilde{\Lambda}_1, \Lambda_2)]$$
$$\text{for} \quad (2-r)\ln 2 < \tilde{\Lambda}_1 < \ln 2$$
$$\simeq \frac{k^2}{b \ln[(1-y_*)/(y_2 - y_*)]} \exp[-kH_2(y_*; \tilde{\Lambda}_1, \Lambda_2)]$$
$$\text{for} \quad \ln 2 \leqslant \tilde{\Lambda}_1 \leqslant \ln 4 \text{ and } y_2 > y_* \qquad \text{(A.9)}$$

where

$$H_2(x; \tilde{\Lambda}_1, \Lambda_2)$$
$$\equiv -\tilde{\Lambda}_1 + \ln 4 + x \ln 2$$
$$- (1-x)\ln(1-x) + (y_2 - x)\ln(y_2 - x) + y_1 \ln y_1$$

Substituting $W_\Lambda(\tilde{\Lambda}_1, \Lambda_2; k, r) \equiv W_1 + W_2$ into (A.1), we get (5.30).

APPENDIX B. SCALING FORM OF $\langle \Lambda_1 \rangle$ NEAR A TRANSITION POINT

For a phase transition of the local Lyapunov exponents in Section 5.2, a scaling form of the mean value of Λ_1 is studied near its transition point.[20,44] We consider the transition between regions B and C so that $z_2 < 0$. The mean of Λ_1 is given by

$$\langle \Lambda_1 \rangle(z_1, z_2; k) = -\frac{1}{k} \lim_{r \to \infty} \frac{\partial}{\partial z_1} \ln \Xi_k(z_1, z_2; r) \tag{B.1}$$

From (5.24), $\langle \Lambda_1 \rangle$ is written as

$$\langle \Lambda_1 \rangle = \ln 4 - \frac{1}{k} \frac{u \ln 2}{g(u)} \frac{\partial}{\partial u} g(u) + O\left(\frac{1}{k}\right) \tag{B.2}$$

where $u \equiv 2^{z_1}[1 + (\kappa_0/\kappa_1)^{z_2}]$ and

$$g(u) = \frac{1 - u^{k-1}}{1 - u} + \frac{(\kappa_0/\kappa_1)^{z_2(k-1)} - u^{k-1}}{(\kappa_0/\kappa_1)^{z_2} - u} \tag{B.3}$$

Since $u = 1$ at the transition point, we assume $0 \leqslant |u - 1| \ll 1$. We also assume $|z_2| > |\ln u|/\ln(\kappa_0/\kappa_1)$, that is, except near the triple point $(-1, 0)$. For $z_2 < 0$ and $\kappa_0 > \kappa_1$, the second term on the lhs of (B.3) is always negligible. Upon putting $y = k \ln u$, $\langle \Lambda_1 \rangle$ turns out to be

$$(\ln 2)\left(1 + \frac{1}{y} - \frac{1}{e^y - 1}\right) + \Delta g \tag{B.4}$$

The second term of (B.4) is always negligible for $|y| < y_0$ such that $1 \ll y_0 \ll k$. Consider the small line segment $(z_1, z_2) = (z_{1c} - t, z_{2c} - at)$, $0 \leqslant |t| \ll 1$, which transversely passes the phase boundary through the transition point (z_{1c}, z_{2c}). As (z_{1c}, z_{2c}) is approached on this segment, $\langle \Lambda_1 \rangle$ takes the following scaling form:

$$\langle \Lambda_1 \rangle(z_{1c} - t, z_{2c} - at; k) = (\ln 2)\left(1 + \frac{1}{y} - \frac{1}{e^y - 1}\right) \quad \text{for} \quad |y| < y_0 \tag{B.5}$$

with

$$y \equiv -kt[a(1 - 2^{z_{1c}}) \ln(\kappa_0/\kappa_1) + \ln 2] \tag{B.6}$$

It is easy to find that $\langle (\Delta \Lambda_1)^2 \rangle \equiv \partial \langle \Lambda_1 \rangle / \partial(kz_1)$, the variance of Λ_1, has a scaling form because Δg of (B.4) is analytic for z_1.

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REFERENCES

1. H. G. Schuster, *Deterministic Chaos* (VCH, Weinheim, 1988); P. Berge, Y. Pomeau, and C. Vidal, *Order within Chaos* (Wiley, New York, 1984).
2. R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, *J. Phys. A* **17**:3521 (1984); U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, N. Ghil, R. Benzi, and G. Parisi, eds. (North-Holland, Amsterdam, 1985).
3. T. C. Halsey, M. J. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, *Phys. Rev. A* **33**:1141 (1986).
4. M. H. Jensen, L. P. Kadanoff, A. Libchaber, I. Procaccia, and J. Stravans, *Phys. Rev. Lett.* **55**:2798 (1985).
5. H. G. E. Hentschel and I. Procaccia, *Physica* **8D**:435 (1983); P. Grassberger, *Phys. Lett.* **97A**:227 (1984).
6. J. P. Eckmann and I. Procaccia, *Phys. Rev. A* **34**:659 (1986); M. Sano, S. Sato, and Y. Sawada, *Prog. Theor. Phys.* **76**:945 (1986).
7. P. Grassberger and I. Procaccia, *Physica* **13D**:34 (1984).
8. H. Fujisaka, *Prog. Theor. Phys.* **70**:1264 (1983); **71**:513 (1984).
9. Y. Takahashi and Y. Oono, *Prog. Theor. Phys.* **71**:851 (1984).
10. H. Fujisaka and M. Inoue, *Prog. Theor. Phys.* **77**:1334 (1987).
11. M. Kohmoto, *Phys. Rev. A* **37**:1345 (1988).
12. P. Grassberger, R. Badii, and A. Politi, *J. Stat. Phys.* **51**:135 (1988).
13. J. P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **57**:617 (1985).
14. T. Morita, H. Hata, H. Mori, T. Horita, and K. Tomita, *Prog. Theor. Phys.* **79**:296 (1987); **78**:511 (1987).
15. G. Paladin and A. Vulpiani, *Phys. Rep.* **156**:147 (1987).
16. T. Yoshida and B. C. So, *Prog. Theor. Phys.* **79**:1 (1988).
17. D. Katzen and I. Procaccia, *Phys. Rev. Lett.* **58**:1169 (1987).
18. T. Bohr and M. H. Jensen, *Phys. Rev. A* **36**:4904 (1987).
19. P. Szepfalusy, T. Tel, A. Csordas, and Z. Kovacs, *Phys. Rev. A* **36**:3525 (1987).
20. N. Mori, T. Kobayashi, H. Hata, T. Morita, T. Horita, and H. Mori, *Prog. Theor. Phys.* **81**:60 (1989).
21. H. Fujisaka, R. Hagihara, and M. Inoue, *Phys. Rev. A* **38**:3680 (1988).
22. T. Bohr and D. Rand, *Physica* **25D**:387 (1987).
23. T. Tel, *Phys. Rev. A* **36**:2507 (1987).
24. M. H. Jensen, L. P. Kadanoff, and I. Procaccia, *Phys. Rev. A* **36**:1409 (1987).
25. H. Hata, T. Horita, H. Mori, T. Morita, and K. Tomita, *Prog. Theor. Phys.* **80**:809 (1988).
26. T. Horita, H. Hata, H. Mori, T. Morita, K. Tomita, S. Kuroki, and H. Okamoto, *Prog. Theor. Phys.* **80**:793 (1988); K. Tomita, H. Hata, T. Horita, H. Mori, and T. Morita, *Prog. Theor. Phys.* **80**:953 (1988).
27. H. Hata, T. Horita, H. Mori, T. Morita, and K. Tomita, *Prog. Theor. Phys.* **81**:11 (1989).
28. K. Tomita, H. Hata, T. Horita, H. Mori, and T. Morita, *Prog. Theor. Phys.* **81**:1 (1989).
29. C. Grebogi, E. Ott, and J. A. Yorke, *Phys. Rev. A* **37**:1711 (1988).

30. H. Shigematsu, Phase transition in the thermodynamics of a local expansion rate for a fully developed chaos, unpublished.
31. E. Ott, W. Withers, and J. A. Yorke, *J. Stat. Phys.* **36**:687 (1984).
32. R. Badii and A. Politi, *Phys. Rev. A* **35**:1288 (1987).
33. J. Guckenheimer and P. Holmes, *Nonlinear Oscillation, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1983).
34. P. Grassberger, *Phys. Lett.* **107A**:101 (1985).
35. M. J. Feigenbaum, M. H. Jensen, and I. Procaccia, *Phys. Rev. Lett.* **57**:1503 (1986).
36. P. Collet and J. P. Eckmann, *Iterated Maps of the Interval as Dynamical Systems* (Birkhauser, Boston, 1980); R. L. Devaney, *An Introduction to Chaotic Dynamical Systems* (Benjamin, Menlo Park, California, 1985).
37. J. D. Farmer, E. Ott, and J. A. Yorke, *Physica* **7D**:153 (1983).
38. M. J. Feigenbaum, *J. Stat. Phys.* **46**:925, 919 (1987).
39. D. Bessi, G. Paladin, G. Turchetti, and S. Vaienti, *J. Stat. Phys.* **51**:109 (1988).
40. D. Ruelle, *Thermodynamic Formalism*, Vol. 5 of *Encyclopedia of Mathematics and Its Applications*, G. C. Rota, ed. (Addison-Wesley, Massachusetts, 1978); R. Bowen, in *Lecture Notes in Mathematics*, No. 470 (Springer, New York, 1975).
41. R. S. Ellis, *Entropy, Large Deviations, and Statistical Mechanics* (Springer, New York, 1985).
42. D. Ruelle, *Statistical Mechanics* (Benjamin, New York, 1969).
43. F. R. Gantmacher, *The Theory of Matrices*, Vol. 2 (Chelsea, New York, 1964).
44. T. Kobayashi, N. Mori, H. Hata, T. Horita, T. Yoshida, and H. Mori, *Prog. Theor. Phys.* **82**:1 (1989); T. Yamane, T. Yamada, and H. Fujisaka, *Prog. Theor. Phys.* **80**:588 (1988).